

ISOPERIMETRY WITH UPPER MEAN CURVATURE BOUNDS AND SHARP STABILITY ESTIMATES

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ABSTRACT. It was proved by Almgren that among boundaries whose mean curvature is bounded from above, perimeter is uniquely minimized by balls. We obtain sharp stability estimates for Almgren's isoperimetric principle and, as an application, we deduce a sharp description of boundaries with almost constant mean curvature under a total perimeter bound which prevents bubbling.

1. INTRODUCTION

1.1. Overview. Our starting point is Almgren's paper [Alm86], where various optimal isoperimetric theorems, involving generalized surfaces and mappings in arbitrary codimension, are introduced. The main goal of [Alm86] is proving the Euclidean isoperimetric inequality in higher codimension. Omitting to specify the crucial point of what is meant by "area minimization with fixed boundary", this is the statement: if S is a n -dimensional compact surface without boundary in \mathbb{R}^{n+k} , $k \geq 1$, and Ω_S is any $(n+1)$ -dimensional area minimizing surface spanned by S , then

$$\frac{\mathcal{H}^n(S)}{\mathcal{H}^{n+1}(\Omega_S)^{n/(n+1)}} \geq \frac{\mathcal{H}^n(D)}{\mathcal{H}^{n+1}(\Omega_D)^{n/(n+1)}} \quad (1.1)$$

where \mathcal{H}^m is the m -dimensional Hausdorff measure in \mathbb{R}^{n+k} , D is a unit radius n -dimensional sphere in \mathbb{R}^{n+k} , and thus Ω_D is a unit radius $(n+1)$ -dimensional ball in \mathbb{R}^{n+k} .

Almgren's proof of (1.1) roughly goes as follows. Assume that S minimizes the left-hand side of (1.1) among boundary-less surfaces enclosing a minimal $(n+1)$ -area equal to $\mathcal{H}^{n+1}(\Omega_D)$. By a first variation argument one finds $|\vec{H}_S| \leq n$, where \vec{H}_S denotes the mean curvature vector of S (with the convention that $|\vec{H}_D| = n$ for the unit n -sphere D). The proof is then completed by proving (see below for more details on this point) the following isoperimetric principle: if S is a boundary-less surface with $|\vec{H}_S| \leq n$, then $\mathcal{H}^n(S) \geq \mathcal{H}^n(D)$. This last fact is what we call here *Almgren's isoperimetric principle*.

The goal of our paper is addressing the stability of Almgren's isoperimetric principle in the codimension one case $k = 1$. This case is relevant in the study of hypersurfaces with almost constant mean curvature, which, as discussed below, is in turn motivated by applications to capillarity theory and geometric flows. We obtain a sharp stability analysis of Almgren's principle, and we deduce from it new sharp results on hypersurfaces with almost constant mean curvature.

It is now convenient to restate Almgren's principle for smooth codimension one boundaries by taking advantage of the fact that, when $k = 1$, Plateau's problem is trivial (as each boundary-less hypersurface S is the boundary of just one set Ω_S of finite volume): *if Ω is a (non-empty) bounded open set with smooth boundary in \mathbb{R}^{n+1} and H_Ω denotes the mean curvature of $\partial\Omega$ (computed with respect to the outer unit normal ν_Ω to Ω), then*

$$H_\Omega(x) \leq n \quad \forall x \in \partial\Omega \quad \text{implies} \quad P(\Omega) \geq P(B_1), \quad (1.2)$$

with $P(\Omega) = P(B_1)$ if and only if Ω is a unit radius ball. Here $n \geq 1$, $B_r(x) = \{y \in \mathbb{R}^{n+1} : |x - y| < r\}$ (for $x \in \mathbb{R}^{n+1}$ and $r > 0$), $B_1 = B_1(0)$ (so that $H_{B_1} = n$) and $P(\Omega) = \mathcal{H}^n(\partial\Omega)$ is the perimeter of Ω .

Almgren's proof of (1.2) is beautifully simple (and, quite interestingly, very close to the argument used in the theory of fully nonlinear elliptic equations to obtain the fundamental Alexandrov-Bakelman-Pucci estimate; see [CC95]). If A denotes the convex envelope of Ω , then the Gaussian curvature K_A of ∂A is equal to the Jacobian of the outer unit normal map $\nu_A : M \rightarrow \mathbb{S}^n$ (where \mathbb{S}^n is the unit sphere) which in turn is injective by convexity. Hence, by the area formula

$$P(B_1) = \mathcal{H}^n(\mathbb{S}^n) = \int_{\partial A} K_A.$$

Now, K_A is the product of n non-negative principal curvatures, so that by the arithmetic-geometric mean inequality $K_A \leq (H_A/n)^n$; and, actually, $K_A = 0$ outside of the contact set $\partial\Omega \cap \partial A$. Since, by assumption, $H_A = H_\Omega \leq n$ on $\partial A \cap \partial\Omega$,

$$P(B_1) = \int_{\partial A \cap \partial\Omega} K_A \leq \int_{\partial A \cap \partial\Omega} \left(\frac{H_A}{n}\right)^n \leq \mathcal{H}^n(\partial A \cap \partial\Omega) \leq \mathcal{H}^n(\partial\Omega) = P(\Omega).$$

We have thus proved that

$$P(\Omega) - P(B_1) = \mathcal{H}^n(\partial\Omega \setminus \partial A) + \underbrace{\int_{\partial A \cap \partial\Omega} \left(1 - \left(\frac{H_\Omega}{n}\right)^n\right)}_{\geq 0 \text{ as } H_\Omega \leq n \text{ on } \partial\Omega} + \underbrace{\int_{\partial A \cap \partial\Omega} \left(\left(\frac{H_A}{n}\right)^n - K_A\right)}_{\geq 0 \text{ by } A \text{ convex, a.-g. mean inequality}} \quad (1.3)$$

which clearly implies (1.2).

Identity (1.3) is the starting point for discussion the rigidity assertion that, if $H_\Omega \leq n$ and $P(\Omega) = P(B_1)$, then Ω is a ball. Indeed, by combining $H_\Omega \leq n$ and $P(\Omega) = P(B_1)$ into (1.3) we find that Ω is convex and that $\partial\Omega$ has constant mean curvature (equal to n) and it is umbilical at each of its points.

Each one of the last two properties individually implies that $\Omega = B_1(x)$ for some $x \in \mathbb{R}^{n+1}$, in the first case thanks to Alexandrov's theorem (see (1.8) below), and in the second case thanks to the *Nabelpunktsatz*:

$$\partial\Omega \text{ is umbilical at each point if and only if } \Omega = B_r(x) \text{ for some } r > 0 \text{ and } x \in \mathbb{R}^{n+1}. \quad (1.4)$$

A third way of deducing from (1.3) that if $H_\Omega \leq n$ and $P(\Omega) = P(B_1)$, then Ω is a unit radius ball, is by exploiting the Euclidean isoperimetric inequality (see (1.5) below). Indeed, (1.3) implies $H_\Omega = n$ on $\partial\Omega$, and then by the divergence theorem and by the tangential divergence theorem one finds

$$\begin{aligned} (n+1)|\Omega| &= \int_{\Omega} \operatorname{div} x = \int_{\partial\Omega} x \cdot \nu_\Omega = \frac{1}{n} \int_{\partial\Omega} H_\Omega x \cdot \nu_\Omega \\ &= \frac{1}{n} \int_{\partial\Omega} \operatorname{div}^{\partial\Omega}(x) = \mathcal{H}^n(\partial\Omega) = P(\Omega) = P(B_1) = (n+1)|B_1| \end{aligned}$$

that is, $|\Omega| = |B_1|$ (here $|\Omega| = \mathcal{H}^{n+1}(\Omega)$ is the volume of Ω). This last information combined with $P(\Omega) = P(B_1)$ says that Ω is an equality case in (1.5), and thus that $\Omega = B_1(x)$ for some $x \in \mathbb{R}^{n+1}$.

We aim at obtaining sharp stability estimates for the isoperimetric principle (1.2). This is achieved in Theorem 1.1, Theorem 1.2 and Theorem 1.5 below, where the structure of sets with small $P(\Omega) - P(B_1)$ is fully described and sharply quantified in terms of various notions of distance of Ω from being a ball. As a by-product we obtain a new sharp stability result for Alexandrov's theorem, concerning the quantitative description of boundaries with almost-constant mean curvature: see Theorem 1.8 and Theorem 1.10 below.

The rest of this introduction is organized as follows. In section 1.2 we recall some stability results for related isoperimetric principles, which serves to illustrate the context of our main theorems. In section 1.3 we state our main stability theorems for Almgren's isoperimetric principle, while in section 1.4 we discuss the application to Alexandrov's theorem. Finally, in section 1.5, we address the organization of the paper.

1.2. Stability theory for related isoperimetric principles. As noticed above, the characterization of equality cases in Almgren's principle can be addressed by exploiting either the Euclidean isoperimetric inequality, Alexandrov's theorem or the *Nabelpunktsatz*. A presentation of some of the various stability theorems that have been obtained for these three isoperimetric principles is a necessary premise to the statement of our main results. We shall discuss in detail the situation for the Euclidean isoperimetric inequality and for Alexandrov's theorem, since Almgren's isoperimetric principle is sitting, so to say, in between these two theorems (see Remark 1.7). The *Nabelpunktsatz* also has a stability theory with sharp and non-sharp results, for which we refer readers to the seminal papers [DLM05, DLM06] in the two-dimensional case, and to [Per11] for additional results in higher dimension.

Let us recall that given a Borel set $\Omega \subset \mathbb{R}^{n+1}$ with finite and positive volume, the *Euclidean isoperimetric inequality* says

$$P(\Omega) \geq P(B_1) \left(\frac{|\Omega|}{|B_1|} \right)^{n/(n+1)}, \quad (1.5)$$

where equality holds if and only if $\Omega = B_r(x)$ for some $r > 0$ and $x \in \mathbb{R}^{n+1}$. (In this generality, $P(\Omega)$ denotes the distributional perimeter of Ω .) A sharp stability estimate for (1.5) is the improved isoperimetric inequality

$$P(\Omega) \geq P(B_1) \left(\frac{|\Omega|}{|B_1|} \right)^{n/(n+1)} \left\{ 1 + c(n) \alpha(\Omega)^2 \right\} \quad (1.6)$$

where $c(n) > 0$ and $\alpha(\Omega)$ denotes the *Fraenkel asymmetry* of Ω , defined as

$$\alpha(\Omega) = \inf \left\{ \frac{|\Omega \Delta B_r(x)|}{|\Omega|} : |B_r(x)| = |\Omega|, x \in \mathbb{R}^{n+1} \right\};$$

see [FMP08, Mag08, FMP10, CL12]. Inequality (1.6) is sharp in the sense that no function of $\alpha(\Omega)$ converging to 0 more slowly than $\alpha(\Omega)^2$ can appear on the right hand side of (1.6). When considering some *a priori* geometric bound on Ω one can obtain a qualitatively stronger information than a control on $\alpha(\Omega)$. This kind of result is more conveniently stated by introducing the *isoperimetric deficit* of Ω

$$\delta_{\text{iso}}(\Omega) = \frac{P(\Omega) |B_1|^{n/(n+1)}}{P(B_1) |\Omega|^{n/(n+1)}} - 1$$

(a non-negative, scale invariant quantity which vanishes if and only if Ω is a ball), in terms of which (1.6) takes the form

$$\delta_{\text{iso}}(\Omega) \geq c(n) \alpha(\Omega)^2.$$

We also recall the further improvement appeared in [FJ11], namely

$$\delta_{\text{iso}}(\Omega) \geq c(n) \left\{ \alpha(\Omega)^2 + \min_{x_0 \in \mathbb{R}^{n+1}} \int_{\partial\Omega} \left| \nu_\Omega(x) - \frac{x - x_0}{|x - x_0|} \right|^2 d\mathcal{H}_x^n \right\}.$$

Denoting by hd the Hausdorff distance between compact subsets of \mathbb{R}^{n+1} , we introduce the *Hausdorff asymmetry* of Ω

$$\text{hd}_\alpha(\Omega) := \inf \left\{ \frac{\text{hd}(\partial\Omega, \partial B_r(x))}{r} : |B_r(x)| = |\Omega|, x \in \mathbb{R}^{n+1} \right\},$$

and then recall the main result from [Fug89]: if Ω is a convex set with $\delta(\Omega) \leq \varepsilon$ for a suitable ε depending on n only, then

$$c(n) \operatorname{hd}_\alpha(\Omega) \leq \begin{cases} \delta_{\text{iso}}(\Omega)^{1/2}, & \text{if } n = 1, \\ \delta_{\text{iso}}(\Omega)^{1/2} \log^{1/2}(1/\delta_{\text{iso}}(\Omega)), & \text{if } n = 2, \\ \delta_{\text{iso}}(\Omega)^{1/n}, & \text{if } n \geq 3. \end{cases} \quad (1.7)$$

We notice that inequality (1.7) also holds (with same exponents) whenever Ω satisfies a uniform cone condition [FGP12] or a uniform John's domain condition [RZ12]. For a recent survey on (1.6) and related issues, see [Fus15].

We now discuss some stability results for *Alexandrov's theorem*: if Ω is an open set in \mathbb{R}^{n+1} with boundary of class C^2 , then

$$H_\Omega \text{ is constant if and only if } \Omega = B_r(x) \text{ for some } r > 0 \text{ and } x \in \mathbb{R}^{n+1}. \quad (1.8)$$

The stability problem for Alexandrov's theorem amounts in understanding the geometry of boundaries with almost-constant mean curvature. To this end it is convenient to introduce the positive quantity

$$H_\Omega^0 = \frac{n P(\Omega)}{(n+1)|\Omega|}, \quad (1.9)$$

which has the following property: if there exists $c \in \mathbb{R}$ such that $H_\Omega = c$ on $\partial\Omega$, then $c = H_\Omega^0$. Next, we define the *constant mean curvature deficit* of Ω as

$$\delta_{\text{cmc}}(\Omega) = \left\| \frac{H_\Omega}{H_\Omega^0} - 1 \right\|_{L^\infty(\partial\Omega)}. \quad (1.10)$$

This quantity is scale invariant and by (1.8) it vanishes if and only if Ω is a ball. The use of the L^∞ -norm in the definition of $\delta_{\text{cmc}}(\Omega)$ arises naturally in the study of capillarity theory, see [CM15, Section 1.2]. The consideration of an L^2 -type deficit would be interesting in view of applications to mean curvature flows.

A stability estimate in terms of $\delta_{\text{cmc}}(\Omega)$ has been obtained in [CV15] under the assumption that Ω satisfies an interior/exterior ball condition of radius $\rho > 0$ at each point of its boundary: if $\delta_{\text{cmc}}(\Omega) \leq \delta_0(n, \rho, P(\Omega))$, then

$$\operatorname{hd}_\alpha(\Omega) \leq C(n, \rho, P(\Omega)) \delta_{\text{cmc}}(\Omega). \quad (1.11)$$

The decay rate of $\operatorname{hd}_\alpha(\Omega)$ in terms of $\delta_{\text{cmc}}(\Omega)$ in (1.11) is sharp. This result is obtained by making quantitative the original moving planes argument of Alexandrov, and using some kind of uniform ball condition seems unavoidable to this end. In view of applications to the study of local minimizers or critical points of capillarity-type energies this assumption is too restrictive. Moreover, an important consequence of the uniform ball assumption is that it prevents the observation of bubbling phenomena. Bubbling is observed, for example, by truncating and then smoothly completing unduloids with very thin necks. In this way one can construct sets Ω with $\delta_{\text{cmc}}(\Omega)$ arbitrarily small that are converging to *arrays of tangent balls*, rather than to a single ball. As shown in [CM15] this is actually the only mechanism by which one can construct boundaries with almost constant mean curvature: more precisely, working with a set Ω that has been rescaled to that $H_\Omega^0 = n$, one has that if $L \in \mathbb{N}$, $\tau \in (0, 1)$, and

$$P(\Omega) \leq (L + \tau) P(B_1) \quad \delta_{\text{cmc}}(\Omega) \leq \delta_0$$

then there exists a finite union G of (at most L) tangent unit radius balls such that

$$\max \left\{ |P(\Omega) - P(G)|, |\Omega \Delta G|, \operatorname{hd}(\partial\Omega, \partial G) \right\} \leq C_0 \delta_{\text{cmc}}(\Omega)^\alpha;$$

moreover, denoting by Σ the part of ∂G obtained by removing a finite family of spherical caps, each with diameter bounded by $\delta_{\text{cmc}}(\Omega)^\alpha$, there exists a map $u \in C^1(\Sigma)$ such that

$$S = \{(1 + u(x)) \nu_G(x) : x \in \Sigma\} \subset \partial \Omega, \quad \mathcal{H}^n(\partial \Omega \setminus S) \leq C_0 \delta_{\text{cmc}}(\Omega)^\alpha,$$

and $\|u\|_{C^1(\Sigma)} \leq C_0 \delta_{\text{cmc}}(\Omega)^\alpha$. The constants δ_0 and C_0 depend on L , λ and n only, and $\alpha = O(n^{-p})$ for explicit values of $p \in \mathbb{N}$. This quantitative description of bubbling is not sharp, and an open problem is that to refining it to obtain sharp decay rates.

1.3. Main results. Our first main result is a sharp stability theorem for Almgren's isoperimetric principle (1.2). Here and in the following we set

$$\delta(\Omega) = P(\Omega) - P(B_1)$$

so that $\delta(\Omega) \geq 0$ for every open set with smooth boundary such that $H_\Omega \leq n$ thanks to Almgren's principle.

Theorem 1.1 (Main stability inequality). *For every $n \geq 1$ there exists positive constants $\delta_0(n)$ and $c_0(n)$ with the following property. If $\Omega \subset \mathbb{R}^{n+1}$ is a bounded, open set with smooth boundary such that $H_\Omega(x) \leq n$ for every $x \in \partial \Omega$ and $\delta(\Omega) \leq \delta_0(n)$, then there exists $x \in \mathbb{R}^n$ such that*

$$P(\Omega) \geq P(B_1) + c_0(n) \left\{ |\Omega \Delta B_1(x)| + \inf \left\{ \varepsilon > 0 : \Omega \subset B_{1+\varepsilon}(x) \right\} \right\}. \quad (1.12)$$

Estimate (1.12) says that $\delta(\Omega)$ controls linearly the Fraenkel asymmetry of Ω and “one side” of its Hausdorff asymmetry whenever $\delta(\Omega)$ is small enough. The decay rate is sharp, in the sense that it is not possible to control these quantities by any function of $\delta(\Omega)$ going to zero faster than $\delta(\Omega)$ itself. A simple example showing this is obtained by considering the family of sets $\Omega_t = B_{1+t}$ as $t \rightarrow 0^+$. Moreover, outside of the regime when $\delta(\Omega)$ is small we cannot expect to control the geometry of Ω , and it is not even true that $\alpha(\Omega) = O(\delta(\Omega))$: to see this, pick any bounded smooth set E , set $\Omega = R E$ for R large enough to entail $H_\Omega \leq n$, and then $\alpha(\Omega) = O(|\Omega|) = O(R^{n+1}) = O(P(\Omega)^{(n+1)/n}) = O(\delta(\Omega)^{(n+1)/n})$ as $R \rightarrow \infty$.

We also notice that one cannot hope to obtain a better type of geometric information on the boundary of Ω . A first example showing this, that can be observed already in dimension $n = 1$, is obtained by letting Ω be a unit ball with arbitrarily many tiny holes, whose boundaries have large but negative mean curvature, and whose presence prevents Ω from containing a ball of radius $1 - \varepsilon$ (i.e., the other “side” of the Hausdorff asymmetry estimate does not hold). If $n = 1$ this kind of problem can be avoided by assuming that $\partial \Omega$ is connected, but in dimension $n \geq 2$ one can indeed draw the same conclusions by constructing sets Ω satisfying the assumptions of Theorem 1.1, with $P(\Omega) - P(B_1)$ arbitrarily small, and with arbitrarily long “inner tentacles” of very negative mean curvature; see Figure 1.

These two examples exploit the possibility for H_Ω to be arbitrarily negative. Now there are two important remarks: first, if we assume a lower bound on the mean curvature, in addition to the upper bound $H_\Omega \leq n$, then it is possible to control the full Hausdorff asymmetry with $\delta(\Omega)$; second, given a set Ω with $H_\Omega \leq n$ we can always find a set E with $|H_E| \leq n$ and whose distance from Ω is controlled in terms of $\delta(\Omega)$. In our next result we start providing a complete quantitative description of the geometry of sets with small $\delta(\Omega)$. In particular we show that up to holes and inner tentacles of small perimeter, every such that is a C^1 -small deformation of a unit ball.

Theorem 1.2 (Structure of sets with small deficit). *Let $n \geq 1$ and let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded, open set with smooth boundary such that $H_\Omega(x) \leq n$ for every $x \in \partial \Omega$.*

(i) If $\delta(\Omega) < P(B_1)$, then Ω is connected and there exists a bounded open set Ω^ such that $\partial \Omega^*$ is connected and*

$$\Omega \subset \Omega^* \text{ with } \partial \Omega^* \subset \partial \Omega,$$

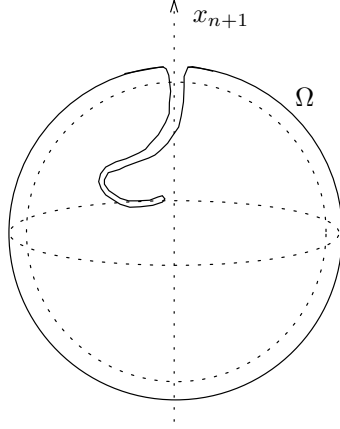


FIGURE 1. If $n \geq 2$, a set $\Omega \subset \mathbb{R}^{n+1}$ with $H_\Omega \leq n$ can have an inner tentacle of length one with small volume and area, and perimeter arbitrarily close to $P(B_1)$. Notice that one needs to start from a ball with radius slightly larger than 1 (and thus with mean curvature slightly smaller than n) to create a tentacle. Indeed, at the opening of the tentacle, Ω turns faster than its reference ball.

$$\begin{aligned} \text{diam}(\Omega) &= \text{diam}(\Omega^*), \\ \mathcal{H}^n(\partial\Omega \setminus \partial\Omega^*) &\leq C(n) \delta(\Omega), \\ |\Omega^* \setminus \Omega| &\leq C(n) \delta(\Omega)^{(n+1)/n}. \end{aligned}$$

(ii) If $\delta(\Omega) \leq \delta_0(n)$, then there exists an open bounded set E with boundary of class $C^{1,1}$ such that

$$\begin{aligned} \Omega &\subset E \\ \text{diam}(\Omega) &= \text{diam}(E), \\ |E \setminus \Omega| + \mathcal{H}^n(\partial E \setminus \partial\Omega) &\leq C(n) \delta(\Omega) \\ \|H_E\|_{L^\infty(\partial E)} &\leq n. \end{aligned}$$

In addition, up to translations,

$$\partial E = \{(1 + u(x))x : x \in \mathbb{S}^n\} \quad (1.13)$$

for some function $u \in C^1(\mathbb{S}^n)$, and for every $\varepsilon > 0$

$$\delta(\Omega) \leq \delta_0(n, \varepsilon) \quad \Rightarrow \quad \|u\|_{C^1(\mathbb{S}^n)} \leq \varepsilon.$$

Remark 1.3. By Theorem 1.2, if Ω has $H_\Omega \leq n$ on $\partial\Omega$ and $\delta(\Omega)$ small, then $\partial\Omega$ has a large connected component $\partial\Omega^*$ which accounts for all the perimeter of Ω up to an error of order $\delta(\Omega)$. In turn, we can chop $\partial\Omega^*$ where its mean curvature is less than $-n$, and complete it into a new set E with bounded mean curvature; see Figure 2. The error we make in doing this is linear in $\delta(\Omega)$ both in volume and perimeter. The new set E is a small C^1 -deformation of the sphere, and Theorem 1.5 below is the sharp stability theorem for this kind of sets. Thus by combining Theorem 1.2 and Theorem 1.5 we shall obtain a complete and sharp analysis of sets with $\delta(\Omega)$ small. Theorem 1.1 will be a direct consequence of these results.

Remark 1.4. Theorem 1.2 requires using a non-classical notion of mean curvature, suitable for boundaries of class $C^{1,1}$. As explained in more detail in section 2 below, for every an open set with $C^{1,1}$ -boundary E there exists a function $H_E \in L^\infty(\mathcal{H}^n \llcorner \partial E)$ such that

$$\int_{\partial E} \text{div}^{\partial E} X \, d\mathcal{H}^n = \int_{\partial E} (X \cdot \nu_E) H_E \, d\mathcal{H}^n \quad \forall X \in C_c^\infty(\mathbb{R}^{n+1}; \mathbb{R}^{n+1}),$$

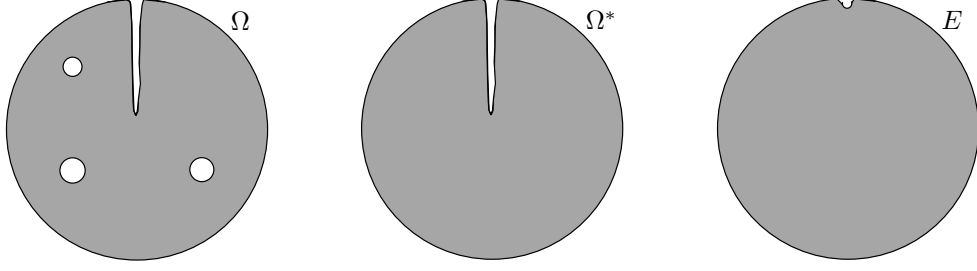


FIGURE 2. Theorem 1.2. A set Ω with small $\delta(\Omega)$ is connected, and all the connected components of its boundary but one have small perimeter. The set Ω^* is obtained by removing them, and may contain (if $n \geq 2$) inner tentacles of order one length. Finally, the set E , which is essentially obtained by truncating the mean curvature H_Ω where $H_\Omega < -n$.

(where $\operatorname{div}^{\partial E} X = \operatorname{div} X - \nu_E \cdot \nabla X[\nu_E]$). The function H_E is the *generalized mean curvature of (the boundary of) E (with respect to the outer unit normal ν_E)*. In the specific case of Theorem 1.2, E is constructed by solving a penalized obstacle problem, see Proposition 3.2, and it will turn out that ∂E is actually analytic, with constant mean curvature equal to $-n$, on $\partial E \setminus \partial\Omega$. The construction of Ω^* in statement (i) is, technically speaking, much simpler, as it is just based on the repeated application of Almgren's principle to the connected components of $\partial\Omega$. From the formal point of view we shall just need part (ii) in the proof of Theorem 1.1, and part (i) has just been included because it is based on an explicit bound on $\delta(\Omega)$, and its proof is based on a very natural idea.

In order to complete the quantitative description of sets with small $\delta(\Omega)$ we are left to quantify the size of the function u appearing in (1.13). This is done in the next theorem.

Theorem 1.5 (Stability of normal perturbations of \mathbb{S}^n). *Let $n \geq 1$, let Ω be the open bounded set with smooth boundary in \mathbb{R}^{n+1} with $H_\Omega \leq n$ \mathcal{H}^n -a.e. on $\partial\Omega$, such that*

$$\int_{\partial\Omega} x d\mathcal{H}_x^n = 0 \quad \partial\Omega = \{(1 + u(x))x : x \in \mathbb{S}^n\},$$

for a function $u : \mathbb{S}^n \rightarrow \mathbb{R}$ such that

$$\|u\|_{C^1(\mathbb{S}^n)} \leq \varepsilon(n).$$

If $\varepsilon(n)$ is suitably small, then

$$\frac{\delta(\Omega)}{C(n)} \leq \int_{\mathbb{S}^n} u \leq C(n) \delta(\Omega) \tag{1.14}$$

and

$$\|u\|_{W^{1,1}(\mathbb{S}^n)} \leq C(n) \delta(\Omega), \tag{1.15}$$

$$\|u^+\|_{C^0(\mathbb{S}^n)} \leq C(n) \delta(\Omega), \tag{1.16}$$

$$\|u\|_{C^0(\mathbb{S}^n)} \leq C(n) \begin{cases} \delta(\Omega), & \text{if } n = 1, \\ \delta(\Omega) \log \left(\frac{C(2)}{\delta(\Omega)} \right), & \text{if } n = 2, \\ \delta(\Omega)^{1/(n-1)}, & \text{if } n \geq 3. \end{cases} \tag{1.17}$$

$$\|u\|_{W^{1,2}(\mathbb{S}^n)} \leq C(n) \sqrt{\|u\|_{C^0(\mathbb{S}^n)} \delta(\Omega) + \delta(\Omega)^2}. \tag{1.18}$$

All these estimates are sharp.

Remark 1.6. Estimate (1.15), (1.16) and (1.17) can be read in more geometric by taking into account that

$$\begin{aligned} |\Omega \Delta B_1| &\leq C(n) \|u\|_{L^1(\Omega)} \\ \inf \{ \varepsilon > 0 : \Omega \subset B_{1+\varepsilon} \} &\leq C(n) \|u^+\|_{C^0(\mathbb{S}^n)} \\ \text{hd}(\partial\Omega, \mathbb{S}^n) &\leq C(n) \|u\|_{C^0(\mathbb{S}^n)}. \end{aligned}$$

Remark 1.7. It seems useful to illustrate the links and differences between the stability problems for the isoperimetric inequality, Alexandrov's theorem, and Almgren's isoperimetric principle. Consider the functional $F(\Omega)$ on sets with finite perimeter with positive and finite volume $\Omega \subset \mathbb{R}^{n+1}$ defined by

$$F(\Omega) = \frac{P(\Omega)}{|\Omega|^{n/(n+1)}}.$$

The isoperimetric theorem and Alexandrov's theorem say that the only global minimizers of F are its only critical points, namely balls in \mathbb{R}^{n+1} . Let Σ denote the set of all balls in \mathbb{R}^{n+1} . Stability for the isoperimetric inequality means controlling the distance of Ω from Σ in terms of the deviation of $F(\Omega)$ from the minimum value of F . Stability for Alexandrov's theorem means controlling the distance of Ω from Σ in terms of the size of δF , the first variation of F . In this second stability problem a complication is due to the presence of "critical points at infinity" (here we are borrowing some terminology from the Yamabe problem, see [Bah89]): precisely, arrays of almost tangent balls with equal radii connected by short necks provide families of almost critical points to F . Stability for Almgren's isoperimetric principle means controlling, under a unilateral constraint on δF , the distance of Ω from Σ in terms of the deviation of F from its minimum value on the constrained class.

In each problem we permit different classes of variations of balls. Consider for example variations of the form $\partial\Omega = \{(1 + u(x))x : x \in \mathbb{S}^n\}$ for some $u \in C^1(\mathbb{S}^n)$ with small C^1 -norm. Taking u to be constant correspond to scaling, so the average of u (projection of u on constants) tells how much we are scaling \mathbb{S}^n when deforming it into $\partial\Omega$. For the isoperimetric inequality, minimizing perimeter with a fixed volume constraint means that the effect of scaling must be negligible, that is $\int_{\mathbb{S}^n} u = O(\int_{\mathbb{S}^n} u^2)$. By contrast, for Almgren's isoperimetric principle, the only constraint on $\int_{\mathbb{S}^n} u$ is on its sign (which must be non-negative, as one must scale outward in order to preserve the condition $H_\Omega \leq n$) but not on its size. For Alexandrov's theorem we have no sign restriction, and $|\int_{\mathbb{S}^n} u|$ is just controlled by the oscillation of the mean curvature from the constant value $n = H_{B_1}$.

With all this in mind, it seems unlikely that one can directly address stability for Almgren's isoperimetric principle from stability for the isoperimetric inequality or for Alexandrov's theorem. It is however possible to use stability for Almgren's isoperimetric principle to understand stability for Alexandrov's theorem, as we illustrate in the next section.

1.4. A sharp estimate for boundaries with almost constant mean curvature. Here we introduce a sharp stability result for Alexandrov's theorem. We address the issue under a global assumption aimed at preventing bubbling, as opposed to the local assumption of a uniform exterior/interior ball condition considered in [CV15]. The assumption we make is that our sets Ω , after setting $H_\Omega^0 = n$ by scaling (recall (1.9)), satisfies $P(\Omega) < 2P(B_1)$. We show then that the constant mean curvature deficit $\delta_{\text{cmc}}(\Omega)$ (defined in (1.10)) controls linearly the Hausdorff asymmetry of Ω . We thus arrive to the same conclusion of (1.11) coming from a different direction.

Theorem 1.8. *If $\tau \in (0, 1)$, $n \geq 1$, Ω is a bounded open set in \mathbb{R}^{n+1} with smooth boundary such that*

$$H_\Omega^0 = n \quad P(\Omega) \leq 2\tau P(B_1) \quad \delta_{\text{cmc}}(\Omega) \leq \delta_0(n, \tau)$$

then there exists $u \in C^{1,1}(\mathbb{S}^n)$ such that, up to a translation, $\partial\Omega = \{(1 + u(x))x : x \in \mathbb{S}^n\}$ with

$$\|u\|_{C^1(\mathbb{S}^n)} \leq C(n) \delta_{\text{cmc}}(\Omega).$$

Remark 1.9. The conclusion of Theorem 1.8 is sharp (think to ellipsoids with small eccentricities) and it implies in particular that

$$\max \left\{ |P(\Omega) - P(B_1)|, \left\| \nu_\Omega - \frac{x}{|x|} \right\|_{C^0(\partial\Omega)}, |\Omega \Delta B_1|, \text{hd}(\partial\Omega, \mathbb{S}^n) \right\} \leq C(n) \delta_{\text{cmc}}(\Omega).$$

As mentioned before, although using an L^∞ -deficit like $\delta_{\text{cmc}}(\Omega)$ is sufficient in view of applications to capillarity theory, having in mind to address convergence to equilibrium in geometric flows (see, for example, [CFM16] for this kind of application of stability theorems to Yamabe-type fast diffusion equations) it would be interesting to obtain a result analogous to Theorem 1.8 with an L^2 -deficit in place of $\delta_{\text{cmc}}(\Omega)$. In fact, without assuming pointwise bounds on the mean curvature of Ω , we can show that the $W^{1,2}$ -distance of $\partial\Omega$ to the unit sphere is bounded linearly in terms of the L^2 -deficit $\|H_\Omega - n\|_{L^2(\partial\Omega)}$ whenever $\partial\Omega$ is a sufficiently C^1 -small perturbation of the unit sphere. Moreover, using slightly stronger integral deficits, we can also control the C^0 -norm of u in terms of the oscillation of the mean curvature.

Theorem 1.10. *If $n \geq 1$ and Ω is an open set with $C^{1,1}$ -boundary such that $\int_{\partial\Omega} x = 0$ and $\partial\Omega = \{(1 + u(x))x : x \in \mathbb{S}^n\}$ for a function $u \in C^{1,1}(\mathbb{S}^n)$ with*

$$\|u\|_{C^1(\mathbb{S}^n)} \leq \varepsilon(n)$$

then

$$\|u\|_{W^{1,2}(\mathbb{S}^n)} \leq C(n) \|H_\Omega - n\|_{L^2(\partial\Omega)}. \quad (1.19)$$

Moreover, if

$$p \geq 2 \quad \text{for } n \leq 3, \quad p > \frac{n}{2} \quad \text{if } n \geq 4. \quad (1.20)$$

then

$$\|u\|_{C^0(\mathbb{S}^n)} \leq C(n, p) \|H_\Omega - n\|_{L^p(\partial\Omega)}. \quad (1.21)$$

Finally, if $\alpha \in (0, 1)$ and $K > 0$ is such that $\|u\|_{C^{1,\alpha}(\mathbb{S}^n)} \leq K$, then

$$\|u\|_{C^{1,\alpha}(\mathbb{S}^n)} \leq C(n, K, \alpha) \delta_{\text{cmc}}(\Omega). \quad (1.22)$$

1.5. Organization of the paper. After introducing some notation and basic facts in section 2, in section 3 we discuss the structure of sets with small Almgren's deficit and prove Theorem 1.2. Section 4 is devoted to the study of normal deformations of \mathbb{S}^n . There we obtain the various estimates from Theorem 1.5 (whose optimality is addressed in section 5) which we use to complete the proof of Theorem 1.1. Finally, the applications to boundaries with almost constant mean curvature is discussed in section 6, where Theorem 1.8 and Theorem 1.10 are proved.

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2. NOTATION AND TERMINOLOGY

Here we gather some definitions and facts that are used throughout the paper. We refer to [Mag12], and point out [Sim83, AFP00, Fed69, KP08, GMS98a, GMS98b] as additional references.

Rectifiable sets and mean curvature: A Borel set $S \subset \mathbb{R}^{n+1}$ is *locally \mathcal{H}^n -rectifiable* in \mathbb{R}^{n+1} if there exists a family of maps $\{f_h\}_{h \in \mathbb{N}} \subset C^1(\mathbb{R}^n; \mathbb{R}^{n+1})$

$$\mathcal{H}^n \left(S \setminus \bigcup_{h \in \mathbb{N}} f_h(\mathbb{R}^n) \right) = 0$$

and $\mathcal{H}^n(S \cap B_R) < \infty$ for every $R > 0$. In particular, $\mathcal{H}^n \llcorner S$ is a Radon measure on \mathbb{R}^{n+1} . If S is locally \mathcal{H}^n -rectifiable, then S admits an *approximate tangent space* $T_x S$ at \mathcal{H}^n -a.e. $x \in S$, that is $T_x S$ is an hyperplane in \mathbb{R}^{n+1} with the property that

$$\lim_{r \rightarrow 0^+} \frac{1}{r^n} \int_S \varphi\left(\frac{y-x}{r}\right) d\mathcal{H}^n(y) = \int_{T_x S} \varphi d\mathcal{H}^n \quad \forall \varphi \in C_c^0(\mathbb{R}^{n+1});$$

see e.g. [Mag12, Theorem 10.2]. If for every such $x \in S$ we denote by $\nu(x)$ a unit normal vector to $T_x S$, then for every $X \in C_c^1(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$ the formula

$$\operatorname{div}^S X(x) = \operatorname{div}(X)(x) - \nabla X(x)[\nu(x)] \cdot \nu(x)$$

defines a Borel map on S . The vector-valued distribution \vec{H}_S

$$\langle \vec{H}_S, X \rangle = \int_S \operatorname{div}^S X d\mathcal{H}^n \quad X \in C_c^1(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$$

is called the *distributional mean curvature* of S . We say that S has *generalized mean curvature* if, given a Borel map $\nu : S \rightarrow \mathbb{S}^n$ such that $\nu(x)$ is normal to $T_x S$ for \mathcal{H}^n -a.e. $x \in S$, there exists $H_S \in L_{\text{loc}}^1(\mathcal{H}^n \llcorner S)$ such that

$$\langle \vec{H}_S, X \rangle = \int_S X \cdot \nu H_S d\mathcal{H}^n \quad \forall X \in C_c^1(\mathbb{R}^{n+1}; \mathbb{R}^{n+1}).$$

Then H_S is the *scalar mean curvature* of S with respect to ν . If we have $H_S \in L^\infty(\mathcal{H}^n \llcorner S)$, then S has *generalized bounded mean curvature*.

Sets of finite perimeter: A Borel set $\Omega \subset \mathbb{R}^{n+1}$ is of finite perimeter in \mathbb{R}^{n+1} if there exists a \mathbb{R}^{n+1} -valued Radon measure μ_Ω on \mathbb{R}^{n+1} such that

$$\int_\Omega \operatorname{div} X(x) dx = \int_{\mathbb{R}^{n+1}} X \cdot d\mu_\Omega \quad \forall X \in C_c^1(\mathbb{R}^{n+1}; \mathbb{R}^{n+1}). \quad (2.1)$$

If $|\mu_\Omega|$ denotes the total variation of μ_Ω , then the set $\partial^* \Omega$ of those $x \in \mathbb{R}^{n+1}$ such that

$$\lim_{r \rightarrow 0^+} \frac{\mu_\Omega(B_r(x))}{|\mu_\Omega|(B_r(x))} \quad (2.2)$$

exists and belongs to \mathbb{S}^n , is called the *reduced boundary* of Ω , and the limit $\nu_\Omega(x) \in \mathbb{S}^n$ in (2.2) is called the *measure-theoretic outer unit normal* to Ω . One always has that $\partial^* \Omega$ is a locally \mathcal{H}^n -rectifiable set and that $T_x(\partial^* \Omega)$ exists for every $x \in \partial^* \Omega$ with $T_x(\partial^* \Omega) = \nu_\Omega(x)^\perp$; moreover, $\mu_\Omega = \nu_\Omega \mathcal{H}^n \llcorner \partial^* \Omega$, so that (2.1) takes the form

$$\int_\Omega \operatorname{div} X(x) dx = \int_{\partial^* \Omega} X \cdot \nu_\Omega d\mathcal{H}^n \quad \forall X \in C_c^1(\mathbb{R}^{n+1}; \mathbb{R}^{n+1});$$

see [Mag12, Chapter 15]. If $|\Omega \Delta \Omega'| = 0$, then clearly $\mu_\Omega = \mu_{\Omega'}$ and $\partial^* \Omega = \partial^* \Omega'$, although the topological boundaries of Ω and Ω' may largely differ. However, up to replace Ω with an Ω' such that $|\Omega \Delta \Omega'| = 0$, it is always possible to obtain

$$\partial \Omega = \{x \in \mathbb{R}^{n+1} : 0 < |\Omega \cap B_r(x)| < |B_r(x)| \quad \forall r > 0\} = \operatorname{spt} \mu_\Omega = \overline{\partial^* \Omega};$$

see [Mag12, Chapter 12]. We shall always assume that our sets of finite perimeter have been normalized so that these identities are in force. Given a Borel set $\Omega \subset \mathbb{R}^{n+1}$ and $t \in [0, 1]$, we set

$$\Omega^{(t)} = \left\{x \in \mathbb{R}^{n+1} : \lim_{r \rightarrow 0^+} \frac{|\Omega \cap B_r(x)|}{|B_r(x)|} = t\right\}$$

for the set of points of density t of Ω . If Ω is a set of locally finite perimeter in \mathbb{R}^{n+1} , then by a result of Federer [Mag12, Theorem 16.2] we have

$$\mathbb{R}^{n+1} \stackrel{\mathcal{H}^n}{=} \Omega^{(0)} \cup \Omega^{(1)} \cup \partial^* \Omega$$

where

$$A \underset{\mathcal{H}^n}{=} B \quad \text{means} \quad \mathcal{H}^n(A \Delta B) = 0.$$

A set of finite perimeter Ω has *generalized mean curvature* H in an open set $A \subset \mathbb{R}^{n+1}$ if $H \in L^1_{\text{loc}}(\mathcal{H}^n \llcorner (A \cap \partial^* \Omega))$ is such that

$$\int_{\partial^* \Omega} \operatorname{div}^{\partial^* \Omega} X \, d\mathcal{H}^n = \int_{\partial^* \Omega} X \cdot \nu_\Omega H \, d\mathcal{H}^n \quad \forall X \in C_c^1(A; \mathbb{R}^{n+1}).$$

In this case, H_Ω is uniquely determined (\mathcal{H}^n -a.e. on $A \cap \partial^* \Omega$) and we set $H = H_\Omega$. Notice that *with this convention* $H_\Omega \geq 0$ for smooth convex sets and $H_\Omega = n$ if Ω is a ball of unit radius. When $H_\Omega \in L^\infty(\mathcal{H}^n \llcorner (A \cap \partial^* \Omega))$ we say that Ω has *generalized bounded mean curvature in A* . An important example to keep in mind in our analysis is the following: if Ω is an open set in \mathbb{R}^{n+1} which, nearby $0 \in \partial\Omega$, is the epigraph in the e_{n+1} -direction of a function $u \in C^{1,1}(D)$ with $u(0) = 0$ and D a ball in \mathbb{R}^n centered at 0 , then Ω has generalized bounded mean curvature in said neighborhood of 0 , and

$$H_\Omega(x + u(x)e_{n+1}) = \frac{\Delta u(x)}{\sqrt{1 + |\nabla u(x)|^2}} - \frac{\nabla^2 u(x)[\nabla u(x)] \cdot \nabla u(x)}{(1 + |\nabla u(x)|^2)^{3/2}} \quad (2.3)$$

for a.e. $x \in D$. Here $\nabla^2 u$ stands the distributional gradient of $u \in C^{1,1}(D) = W^{2,\infty}(D)$, so that $\nabla^2 u(x)$ is indeed uniquely determined a.e. on D .

Perimeter almost-minimizers: We say that $E \subset \mathbb{R}^{n+1}$ is a perimeter (Λ, r_0, α) -minimizer in some open set A if for some $\alpha \in (0, 1)$

$$P(E; W) \leq P(F; W) + \Lambda r^{n+2\alpha}$$

whenever $E \Delta F \subset\subset W \subset\subset A$ with $\operatorname{diam}(W) = r < r_0$. In this context, the classical ε -regularity theorem and dimension reduction scheme lead to the following statement: if E is a (Λ, r_0, α) -minimizer in A , then there exists a set $\Sigma \subset A \cap \partial E$, relatively closed in A , such that E is an open set with $C^{1,\alpha}$ -boundary in $A \setminus \Sigma$ and the Hausdorff dimension of Σ is at most $n - 7$ (actually, Σ is locally finite in A if $n = 7$, and $\Sigma = \emptyset$ if $n \leq 6$); see, e.g. [Tam82, Mag12]. The result is sharp, in the sense that every open set with $C^{1,\alpha}$ -boundary is a (Λ, r_0, α) -minimizer; see [Tam84, Section 1.6]. For the reader's convenience this last fact is recalled in the following proposition, where for $x \in \mathbb{R}^{n+1}$, $\nu \in \mathbb{S}^n$ and $r > 0$ we set

$$\begin{aligned} \mathbf{D}_r^\nu(x) &= \{y \in \mathbb{R}^{n+1} : y = x + z, z \cdot \nu = 0, |z| < r\} \\ \mathbf{C}_r^\nu(x) &= \{y \in \mathbb{R}^{n+1} : y = x + z + t\nu, z \in \mathbf{D}_r^\nu, |t| < r\}. \end{aligned} \quad (2.4)$$

Proposition 2.1. *If Ω is an open set with $C^{1,\alpha}$ -boundary in the open set A , then for every $A' \subset\subset A$ there exist $\Lambda \geq 0$ and $r_0 > 0$ such that Ω is a (Λ, r_0, α) -minimizer in A' .*

Proof of Proposition 2.1. By definition, the fact that Ω is an open set with $C^{1,\alpha}$ -boundary in A means that for every $x_0 \in A \cap \partial\Omega$ there exist $r_0 > 0$, $\nu_0 \in \mathbb{S}^n$ and $u_0 \in C^{1,\alpha}(\mathbf{D}_{2r_0}^{\nu_0})$ such that $u_0(0) = 0$, $\nabla u_0(0) = 0$, $\operatorname{Lip}(u_0) \leq 1$ and

$$\Omega \cap \mathbf{C}_{2r_0}^{\nu_0}(x_0) = \{y \in \mathbb{R}^{n+1} : y = x_0 + z + t\nu_0, z \in \mathbf{D}_{2r_0}^{\nu_0}, u_0(z) < t < 2r_0\}.$$

Since $A' \subset\subset A$ we can consider a same value of r_0 for every $x_0 \in A' \cap \partial\Omega$, and also require that the $4r_0$ -neighborhood of A' is compactly contained in A .

Now let F be such that $F \Delta \Omega \subset\subset W$ for some set $W \subset\subset A'$ with $\operatorname{diam}(W) = s < r_0$. Since we aim to prove

$$P(\Omega; W) \leq P(F; W) + \Lambda s^{n+2\alpha}, \quad (2.5)$$

we can assume that $W \cap \partial\Omega \neq \emptyset$, for otherwise $P(\Omega; W) = 0$. Thus we can find $x_0 \in A' \cap \partial\Omega$ such that

$$W \subset B_s(x_0) \subset\subset \mathbf{C}_r^{\nu_0}(x_0) \quad \forall r > s.$$

Let $r \in (s, 2r_0)$. By applying the divergence theorem on $F \cap \mathbf{C}_r^{\nu_0}(x_0)$ to the vector field $X(x) = \varphi(x) \nu_0$, where $\varphi \in C_c^\infty(A)$ with $0 \leq \varphi \leq 1$ and $\varphi = 1$ on the $4r_0$ -neighborhood of A' , we find that

$$0 = \int_{F \cap \mathbf{C}_r^{\nu_0}(x_0)} \operatorname{div} X = \int_{\partial^*(\mathbf{C}_r^{\nu_0}(x_0) \cap F)} \nu_0 \cdot \nu_{\mathbf{C}_r^{\nu_0}(x_0) \cap F}. \quad (2.6)$$

Now, for a.e. $r \in (s, 2r_0)$ we have

$$\partial^*(\mathbf{C}_r^{\nu_0}(x_0) \cap F) \stackrel{\mathcal{H}^n}{=} (\mathbf{C}_r^{\nu_0}(x_0) \cap \partial^* F) \cup (F^{(1)} \cap \partial \mathbf{C}_r^{\nu_0}(x_0)) \quad (2.7)$$

$$F^{(1)} \cap \partial \mathbf{C}_r^{\nu_0}(x_0) \stackrel{\mathcal{H}^n}{=} \Omega \cap \partial \mathbf{C}_r^{\nu_0}(x_0) \quad (2.8)$$

and

$$\nu_0 \cdot \nu_{\mathbf{C}_r^{\nu_0}(x_0) \cap F}(y) = \begin{cases} \nu_0 \cdot \nu_F(y), & \text{for } \mathcal{H}^n\text{-a.e. } y \in \mathbf{C}_r^{\nu_0}(x_0) \cap \partial^* F \\ 1, & \text{for } \mathcal{H}^n\text{-a.e. } y \in F^{(1)} \cap \mathbf{D}_r^{\nu_0}(x_0 + r\nu_0) \\ 0, & \text{at } \mathcal{H}^n\text{-a.e. other point } y \in \partial^*(\mathbf{C}_r^{\nu_0}(x_0) \cap F) \end{cases}$$

see, e.g., [Mag12, Chapter 16]. Therefore,

$$\int_{\partial^*(\mathbf{C}_r^{\nu_0}(x_0) \cap F)} \varphi \nu_0 \cdot \nu_{\mathbf{C}_r^{\nu_0}(x_0) \cap F} = -\mathcal{H}^n(\Omega \cap \mathbf{D}_r^{\nu_0}(x_0 + r\nu_0)) + \int_{\mathbf{C}_r^{\nu_0}(x_0) \cap \partial^* F} \nu_0 \cdot \nu_F$$

and (2.6) gives

$$P(F; \mathbf{C}_r^{\nu_0}(x_0)) \geq \mathcal{H}^n(\Omega \cap \mathbf{D}_r^{\nu_0}(x_0 + r\nu_0)) = \mathcal{H}^n(\mathbf{D}_r^{\nu_0}).$$

We thus find

$$\begin{aligned} P(\Omega; W) - P(F; W) &= P(\Omega; \mathbf{C}_r^{\nu_0}(x_0)) - P(F; \mathbf{C}_r^{\nu_0}(x_0)) \\ &= \int_{\mathbf{D}_r^{\nu_0}} \sqrt{1 + |\nabla u_0|^2} - P(F; \mathbf{C}_r^{\nu_0}(x_0)) \\ &\leq \int_{\mathbf{D}_r^{\nu_0}} (\sqrt{1 + |\nabla u_0|^2} - 1) \leq C(n) r^n \|\nabla u_0\|_{C^0(\mathbf{D}_r^{\nu_0})}^2. \end{aligned}$$

where in the last step we have used $\sqrt{1 + |\xi|^2} - 1 \leq |\xi|^2/2$ for every $\xi \in \mathbb{R}^n$. Since $\nabla u_0(0) = 0$, we conclude that

$$\|\nabla u_0\|_{C^0(\mathbf{D}_r^{\nu_0})}^2 \leq C r^{2\alpha}$$

for a constant C depending on the α -Hölder semi-norm of ∇u_0 on $\mathbf{D}_r^{\nu_0}$. Combining everything together we have proved

$$P(\Omega; W) \leq P(F; W) + \Lambda r^{n+2\alpha}$$

for a.e. $r \in (s, 2r_0)$. Letting $r \rightarrow s^+$ we conclude the proof of (2.5). \square

We conclude this section with another useful technical remark.

Proposition 2.2. *If E is a (Λ, r_0, α) -minimizer in an open set $A \subset \mathbb{R}^{n+1}$, Σ is the singular set of E in A , and H_E is the generalized mean curvature of E in $A \setminus \Sigma$, then H_E (extended to constantly take the value 0 on Σ , say) is the generalized mean curvature of E in A .*

Proof of Proposition 2.2. This is based on a standard cut-off function and covering argument based on the fact that the Hausdorff dimension of Σ is at most $n-7$ and, by (Λ, r_0, α) -minimality in A , $P(E; B_r(x_0)) \leq C(n, \Lambda, r_0, \alpha) r^n$ for any ball $B_r(x_0) \subset\subset A$. We omit the details. \square

3. STRUCTURE OF SETS WITH SMALL $\delta(\Omega)$ AND REDUCTION TO NORMAL GRAPHS

This section is devoted to discussing the reduction to normal graphs over \mathbb{S}^n in the stability problem for Almgren's isoperimetric principle. There is a first interesting observation, which is based on the simple idea of applying Almgren's principle to each connected component of $\partial\Omega$, and leads to a sharp structural decomposition result under the quite explicit assumption that $\delta(\Omega) < P(B_1)$, or, equivalently, that $P(\Omega) < 2P(B_1)$. This argument is presented in Proposition 3.1 below. This result allows one to focus on the case of a simply connected set Ω with connected boundary. The mean curvature of $\partial\Omega$ is bounded from above, but not from below. This is unavoidable, even with arbitrarily small deficit. However, we can construct a subset E of Ω , whose boundary has bounded mean curvature and largely overlaps with $\partial\Omega$. If $\delta(\Omega)$ is small enough, ∂E will be a normal graph over \mathbb{S}^n , described by a function u with small C^1 norm. For this kind of boundaries we can obtain a sharp stability theory by mixing spectral analysis, elliptic regularity, and interpolation inequalities, see section 4. The construction of E , starting from Ω with small deficit, is discussed in Proposition 3.2 below. It is based on the regularity theories for perimeter almost-minimizers and for free-boundary problems. This result seems to have an independent interest, as it should be applicable to other variational problems where one needs to truncate mean curvature.

3.1. Applying Almgren's principle to the components of a boundary. Here we prove the following proposition, which takes care of the first part of the statement of Theorem 1.2.

Proposition 3.1. *If Ω is an open bounded set in \mathbb{R}^{n+1} with smooth boundary such that $H_\Omega \leq n$ and $\delta(\Omega) < P(B_1)$, then there exists an open bounded connected set Ω^* with smooth, connected boundary such that*

$$|\Omega^* \setminus \Omega| \leq C(n) \delta(\Omega)^{(n+1)/n} \quad \Omega \subset \Omega^*.$$

Moreover, $\partial\Omega^* \subset \partial\Omega$, so that, in particular,

$$H_{\Omega^*} \leq n \text{ and } \delta(\Omega^*) \leq \delta(\Omega).$$

Proof of Proposition 3.1. Let $\{A^j\}_{j \in J}$ be the family of the connected components of Ω . Clearly we can apply (1.2) to each A^j . As a consequence

$$P(\Omega) = \sum_{j \in J} P(A^j) \geq \#J P(B_1)$$

so that if $\delta(\Omega) < P(B_1)$, then $\#J = 1$. In other words, Ω is connected.

Now let $\{S^i\}_{i \in I}$ be the family of the connected components of $\partial\Omega$. Each S^i is a compact, connected, orientable hypersurface in \mathbb{R}^{n+1} such that $S^i = \partial\Omega^i$ for an open set with smooth boundary Ω^i such that $|\Omega^i| < \infty$. Now, by continuity, either $\nu_{\Omega^i} = \nu_\Omega$ on S^i or $\nu_{\Omega^i} = -\nu_\Omega$ on S^i , and accordingly we define a partition $\{I^+, I^-\}$ of I . If $i \in I^+$, then the mean curvature H_{S^i} of S^i computed with respect to ν_{Ω^i} satisfies $H_{S^i} \leq n$ on S^i , and thus by Almgren's isoperimetric principle

$$\mathcal{H}^n(S^i) \geq P(B_1).$$

Since $\delta(\Omega) < P(B_1)$ this means that $\#I^+ \leq 1$. By sliding an hyperplane from infinity until it touches S^i we find that $I^+ \neq \emptyset$, and thus $\#I^+ = 1$. Now, if S^* denotes the only element of $\{S^i\}_{i \in I^+}$, and Ω^* is the bounded open set with finite volume such that $\nu_{\Omega^*} = \nu_\Omega$, then

$$\Omega = \Omega^* \setminus \bigcup_{i \in I^-} \Omega^i \subset \Omega^*,$$

and

$$\delta(\Omega) = P(\Omega^*) - P(B_1) + \sum_{i \in I^-} P(\Omega^i) \geq \sum_{i \in I^-} P(\Omega^i) \geq c(n) \sum_{i \in I^-} |\Omega^i|^{n/(n+1)}$$

so that

$$|\Omega^* \setminus \Omega| = \sum_{i \in I^-} |\Omega^i| \leq C(n) \delta(\Omega)^{(n+1)/n}.$$

Finally, since $\partial\Omega^*$ is connected, we have that Ω^* is connected. \square

3.2. Truncating the mean curvature of a set. The following result is particularly useful in “truncating the mean curvature of a set”. The result itself will probably not be surprising for experts in the obstacle problem, but we have included a detailed proof for the sake of clarity.

Proposition 3.2. *If $\lambda > 0$, $\alpha \in (0, 1)$ and Ω is an open bounded set with $C^{1,\alpha}$ -boundary in \mathbb{R}^{n+1} , then there exist minimizers in the variational problem*

$$\inf \left\{ P(E) + \lambda |E| : \Omega \subset E, |E| < \infty \right\}. \quad (3.1)$$

If E_λ is one such minimizer, then:

- (i) E_λ is contained in the convex envelope of Ω with $\text{diam}(E_\lambda) = \text{diam}(\Omega)$ and

$$\lambda |E_\lambda \setminus \Omega| + \mathcal{H}^n(\partial^* E_\lambda \setminus \partial\Omega) \leq \delta(\Omega). \quad (3.2)$$

- (ii) there exists a closed set $\Sigma \subset \partial E_\lambda$ such that E_λ is an open set with $C^{1,\beta}$ -boundary in $\mathbb{R}^{n+1} \setminus \Sigma$ for some $\beta \in (0, 1)$. In particular, $\mathcal{H}^n(\partial E_\lambda \Delta \partial^* E_\lambda) = 0$.
 (iii) if Ω has $C^{2,\beta}$ -boundary in \mathbb{R}^{n+1} , then E_λ is an open set with $C^{1,1}$ -boundary in $\mathbb{R}^{n+1} \setminus \Sigma$, and E_λ has generalized bounded mean curvature in \mathbb{R}^{n+1} satisfying

$$\|H_{E_\lambda}\|_{L^\infty(\partial E_\lambda)} \leq \max\{\|(H_\Omega)^+\|_{C^0(\partial\Omega)}, \lambda\}. \quad (3.3)$$

Proof of Proposition 3.2. Step one: We prove the existence of E_λ and conclusion (i). First, we notice that the infimum in (3.1) is finite, as Ω itself is a competitor with finite energy. The convex hull A of Ω is bounded, and energy is decreased by intersecting E with A . Thus we can minimize over $E \subset A$, and by standard lower semicontinuity and compactness properties of perimeter, there exists at least a minimizer E_λ . Since $\Omega \subset E_\lambda \subset A$ we have $\text{diam}(E_\lambda) = \text{diam}(\Omega)$. By testing E_λ against Ω we find

$$\lambda |E_\lambda \setminus \Omega| \leq P(\Omega) - P(E_\lambda) = P(\Omega; E_\lambda^{(1)}) - \mathcal{H}^n(\partial^* E_\lambda \setminus \partial\Omega),$$

where we have used $P(\Omega) = P(\Omega; E_\lambda^{(1)}) + \mathcal{H}^n(\partial\Omega \cap \partial^* E_\lambda)$. By $E_\lambda \subset A$ we find $E_\lambda^{(1)} \subset A^{(1)}$, and thus

$$P(\Omega; E_\lambda^{(1)}) \leq P(\Omega; A^{(1)}) = P(\Omega) - \mathcal{H}^n(\partial\Omega \cap \partial A) = \mathcal{H}^n(\partial\Omega \setminus \partial A) \leq \delta(\Omega),$$

thanks to (1.3). This proves (3.2).

Step two: We prove conclusion (ii). By Proposition 2.1 there exist positive constants r_0 and Λ such that

$$P(\Omega; V) \leq P(H; V) + \Lambda r^{n+2\alpha} \quad (3.4)$$

whenever $H \Delta \Omega \subset \subset W$ with $\text{diam}(W) = r < r_0$. Let us consider a set F such that $F \Delta E_\lambda \subset \subset W$ for a bounded open set W , and set $r = \text{diam}(W) < r_0$. Let us assume first that

$$\mathcal{H}^n(W \cap \partial\Omega) = \mathcal{H}^n(W \cap \partial E_\lambda) = 0. \quad (3.5)$$

If we set $H = F \cap \Omega$, then we have that

$$H \Delta \Omega = \Omega \setminus (F \cap \Omega) = \Omega \setminus F \subset E_\lambda \setminus F \subset \subset W$$

so that if $r < r_0$, then by (3.4)

$$P(\Omega; W) \leq P(F \cap \Omega; W) + \Lambda r^{n+2\alpha}.$$

Since $F \cap \Omega \cap W^c = E_\lambda \cap \Omega \cap W^c = \Omega \cap W^c$, thanks to (3.5) we actually have

$$P(F \cap \Omega) - P(\Omega) = P(F \cap \Omega; W) - P(\Omega; W)$$

and we have thus obtained

$$P(\Omega) \leq P(F \cap \Omega) + \Lambda r^{n+2\alpha}. \quad (3.6)$$

At the same time $F \cup \Omega$ is admissible in (3.1), thus by the general inequality

$$P(N \cap M) + P(N \cup M) \leq P(N) + P(M) \quad \forall N, M \subset \mathbb{R}^{n+1},$$

we get

$$P(E_\lambda) \leq P(F) + P(\Omega) - P(F \cap \Omega) + \lambda |E_\lambda \Delta (F \cup \Omega)|. \quad (3.7)$$

By $F \Delta E_\lambda \subset \subset W$ we have $P(F) - P(E_\lambda) = P(F; W) - P(E_\lambda; W)$, while $E_\lambda \Delta (F \cup \Omega) \subset F \Delta E_\lambda \subset \subset W$ gives us $|E_\lambda \Delta (F \cup \Omega)| \leq C(n) r^{n+1}$, so that (3.6) and (3.7) imply

$$P(E_\lambda; W) \leq P(F; W) + C(n, \lambda) r^{n+1} + \Lambda r^{n+2\alpha}. \quad (3.8)$$

We have thus proved that E_λ is a $(\Lambda', r_0, \min\{1/2, \alpha\})$ -minimizer in \mathbb{R}^{n+1} . As a consequence there exists a closed set $\Sigma \subset \partial E_\lambda$ such that E_λ is an open set with $C^{1,\beta}$ -boundary on $\mathbb{R}^{n+1} \setminus \Sigma$ for $\beta = \min\{1/2, \alpha\}$.

Step three: We prove statement (iii). By a first variation argument based on the minimality of E_λ in (3.1) one finds that

$$\int_{\partial E_\lambda} \operatorname{div}^{\partial E_\lambda} X \geq -\lambda \int_{\partial E_\lambda} X \cdot \nu_{E_\lambda} \quad (3.9)$$

for every $X \in C_c^\infty(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$ with $X \cdot \nu_{E_\lambda} \geq 0$ on ∂E_λ : that is, $H_{E_\lambda} \geq -\lambda$ on ∂E_λ in distributional sense. More precise information is found by considering the open sets

$$\begin{aligned} A_1 &= \{x \in \mathbb{R}^{n+1} : \exists r > 0 \text{ s.t. } B_r(x) \cap \partial E_\lambda \subset \partial \Omega\} \\ A_2 &= \{x \in \mathbb{R}^{n+1} \setminus \Sigma_\lambda : \exists r > 0 \text{ s.t. } B_r(x) \cap \Omega = \emptyset\} = \mathbb{R}^{n+1} \setminus (\Sigma_\lambda \cup \overline{\Omega}) \end{aligned}$$

Since Ω has generalized bounded mean curvature in \mathbb{R}^{n+1} (recall the discussion around (2.3)), we find that E_λ has generalized bounded mean curvature in A_1 satisfying

$$H_{E_\lambda} = H_\Omega \quad \text{on } A_1 \cap \partial E_\lambda. \quad (3.10)$$

Moreover, again by a first variation argument based on its minimality in (3.1), E_λ has generalized bounded mean curvature in A_2 given by

$$H_{E_\lambda} = -\lambda \quad \text{on } A_2 \cap \partial E_\lambda. \quad (3.11)$$

(In particular, E_λ has smooth boundary in $A_1 \cup A_2$.) We now pick

$$x_0 \in \partial E_\lambda \setminus (\Sigma \cup A_1 \cup A_2). \quad (3.12)$$

and claims that there exists $\rho > 0$ such that E_λ has $C^{1,1}$ -boundary in $B_\rho(x_0)$ with

$$\|H_{E_\lambda}\|_{L^\infty(B_\rho(x_0) \cap \partial E_\lambda)} \leq \max\{\lambda, \|(H_\Omega)^+\|_{C^0(B_\rho(x_0) \cap \partial \Omega)}\}.$$

By combining this claim with (3.10) and (3.11), we shall conclude that E_λ has generalized bounded mean curvature in $\mathbb{R}^{n+1} \setminus \Sigma$. Thanks to Proposition 2.2 this last fact will complete the proof of step three.

Given x_0 as in (3.12), since $x_0 \notin \Sigma$, up to a translation and a rotation (so that $x_0 = 0$ and $\nu_{E_\lambda}(0) = -e_{n+1}$) we have that there exist $r > 0$ and

$$\begin{aligned} u &\in C^{1,1}(\mathbb{R}^n), & u(0) &= 0, & \nabla u(0) &= 0, & \operatorname{Lip}(u) &\leq 1, \\ v &\in C^{1,\beta}(\mathbb{R}^n), & v(0) &= 0, & \nabla v(0) &= 0, & \operatorname{Lip}(v) &\leq 1, \end{aligned}$$

such that, setting

$$\mathbf{D}_r(x) = \mathbf{D}_r^{e_{n+1}}(x) \quad \mathbf{D}_r = \mathbf{D}_r(0) \quad \mathbf{C}_r(x) = \mathbf{C}_r^{e_{n+1}}(x) \quad \mathbf{C}_r = \mathbf{C}_r(0),$$

(see (2.4) for the notation used here) then we have

$$\Omega \cap \mathbf{C}_r = \{(x, x_{n+1}) \in \mathbf{C}_r : x_{n+1} < u(x)\}$$

$$\begin{aligned}
E_\lambda \cap \mathbf{C}_r &= \{(x, x_{n+1}) \in \mathbf{C}_r : x_{n+1} < v(x)\} \\
(\partial E_\lambda \setminus \partial \Omega) \cap \mathbf{C}_r &= \{(x, v(x)) \in \mathbf{C}_r : v(x) > u(x)\} \\
\partial E_\lambda \cap \partial \Omega \cap \mathbf{C}_r &= \{(x, v(x)) \in \mathbf{C}_r : v(x) = u(x)\}.
\end{aligned}$$

Since $\Omega \subset E_\lambda$ we have $u \leq v$ on \mathbf{D}_r , where thanks to (3.10), (3.11) and (3.9) it holds

$$-\operatorname{div} \left(\frac{\nabla v}{\sqrt{1 + |\nabla v|^2}} \right) \geq -\lambda \quad \text{weakly on } \mathbf{D}_r \quad (3.13)$$

$$-\operatorname{div} \left(\frac{\nabla v}{\sqrt{1 + |\nabla v|^2}} \right) = -\lambda \quad \text{strongly on } \mathbf{D}_r \cap \{v < u\} \quad (3.14)$$

$$-\operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = h \quad \text{strongly } \mathbf{D}_r. \quad (3.15)$$

Here, $h(x) = H_\Omega(x + u(x)e_{n+1})$ for every $x \in \mathbf{D}_r$. We now claim that there exist

$$s_0 = s_0(n, \lambda, r, \|\nabla^2 u\|_{C^0(\mathbf{D}_r)}) \in (0, r/4) \quad C_0 = C_0(n, \lambda, r, \|\nabla^2 u\|_{C^0(\mathbf{D}_r)}) \quad (3.16)$$

such that

$$\sup_{\mathbf{D}_s(y)} (v - u) \leq C_0 s^2 \quad \forall s \in (0, s_0), y \in \mathbf{D}_{s_0} \cap \{u = v\}. \quad (3.17)$$

We prove this by a classical barrier argument from the regularity theory of obstacle problems, see [Caf98, Theorem 2, Lemma 2]. Our barriers will be given by spherical caps. Let us fix y as in (3.17), and set

$$\xi(y) = y - \frac{n}{\lambda} \frac{\nabla u(y)}{\sqrt{1 + |\nabla u(y)|^2}} \quad (3.18)$$

$$\psi_y(x) = u(y) - \sqrt{(n/\lambda)^2 - |x - \xi(y)|^2} + \sqrt{(n/\lambda)^2 - |y - \xi(y)|^2}, \quad x \in \mathbb{R}^n \quad (3.19)$$

so that the graph of ψ_y over $\mathbf{D}_{n/\lambda}(\xi(y))$ is a half-sphere of radius $1/\lambda$, which is tangent to the graph of u at the point $y + u(y)e_{n+1}$ thanks to (3.18): in particular

$$\psi_y(y) = u(y) \quad \nabla \psi_y(y) = \nabla u(y), \quad (3.20)$$

$$-\operatorname{div} \left(\frac{\nabla \psi_y}{\sqrt{1 + |\nabla \psi_y|^2}} \right) = -\lambda \quad \text{on } \mathbf{D}_{n/\lambda}(\xi(y)). \quad (3.21)$$

Notice that, thanks to (3.16), we can entail

$$\mathbf{D}_{s_0}(y) \subset \mathbf{D}_{n/2\lambda}(\xi(y)) \cap \mathbf{D}_r. \quad (3.22)$$

Indeed if $x \in \mathbf{D}_{s_0}(y)$, then by $y \in \mathbf{D}_{s_0}$, $\nabla u(0) = 0$ and (3.16) (with s_0 suitably small) we have

$$|x - \xi(y)| \leq s_0 + |y - \xi(y)| \leq s_0 + \frac{n}{\lambda} |\nabla u(y)| \leq s_0 + \frac{n}{\lambda} \|\nabla^2 u\|_{C^0(\mathbf{D}_r)} s_0 < \frac{n}{2\lambda}.$$

This guarantees that ψ_y is well-defined on $\mathbf{D}_{s_0}(y) \subset \mathbf{D}_r$. We also observe that for every $s \in (0, s_0)$,

$$\psi_y - C_0 s^2 \leq v \quad \text{on } \mathbf{D}_s(y). \quad (3.23)$$

Indeed if $x \in \mathbf{D}_s(y)$ with $s \in (0, s_0)$, then by (3.22) and (3.20)

$$\begin{aligned}
v(x) &\geq u(x) \geq u(y) + \nabla u(y) \cdot (x - y) - \|\nabla^2 u\|_{C^0(\mathbf{D}_r)} \frac{s^2}{2} \\
&= \psi_y(y) + \nabla \psi_y(y) \cdot (x - y) - \|\nabla^2 u\|_{C^0(\mathbf{D}_r)} \frac{s^2}{2} \\
&\geq \psi_y(x) - C_0 s^2
\end{aligned}$$

since $x \in \mathbf{D}_{s_0}(y) \subset \mathbf{D}_{n/2\lambda}(\xi(y))$, where

$$\|\nabla^2 \psi_y\|_{C^0(\mathbf{D}_{n/2\lambda}(\xi(y)))} \leq C(n, \lambda).$$

Thanks to (3.23), if we set

$$w = v - \psi_y + C_0 s^2 \quad \text{on } \mathbf{D}_s(y)$$

then $w \geq 0$ on $\mathbf{D}_{s_0}(y)$. By (3.13) and (3.21), there exists a matrix-field $A \in C^{0,\beta}(\mathbf{D}_{s_0}(y); \mathbb{R}_{\text{sym}}^{n \times n})$ with

$$\|A\|_{C^{0,\beta}(\mathbf{D}_{s_0}(y))} \leq K \quad \frac{\text{Id}}{K} \leq A(x) \leq \text{Id}, \quad \forall x \in \mathbf{D}_{s_0}(y),$$

(where here and in the following K denotes a generic positive constant depending on $n, \lambda, r, [\nabla v]_{C^{0,\beta}(\mathbf{D}_r)}$ and $\|\nabla^2 u\|_{C^0(\mathbf{D}_r)}$) such that

$$\text{div}(A \nabla w) \leq 0 \quad \text{weakly on } \mathbf{D}_{s_0}(y); \quad (3.24)$$

and, thanks to (3.14),

$$\text{div}(A \nabla w) = 0 \quad \text{strongly on } \mathbf{D}_{s_0}(y) \cap \{u > v\}. \quad (3.25)$$

Let w_1 be the solution to

$$\begin{cases} \text{div}(A \nabla w_1) = 0 & \text{in } \mathbf{D}_s(y), \\ w_1 = w & \text{on } \partial \mathbf{D}_s(y). \end{cases} \quad (3.26)$$

By the weak maximum principle, (3.24), (3.26), and $w \geq 0$ on $\mathbf{D}_{s_0}(y)$,

$$0 \leq w_1 \leq w \quad \text{on } \mathbf{D}_s(y).$$

By the Harnack inequality, $w_1 \leq w$, and $u(y) = v(y) = \psi(y)$, for every $s \in (0, s_0)$ we have

$$\sup_{\mathbf{D}_s(y)} w_1 \leq K w_1(y) \leq K w(y) = K (v(y) - \psi(y) + C_0 s^2) \leq K s^2.$$

By the strong maximum principle, (3.25), and (3.26), $w_2 = w - w_1$ attains its maximum over the closure of $\mathbf{D}_s(y) \cap \{u < v\}$ at some point $x_1 \in \mathbf{D}_s(y) \cap \{u = v\}$. By $w_2 \leq w$, $u(x_1) = v(x_1)$, $u(y) = \psi(y)$, and $\nabla u(y) = \nabla \psi(y)$,

$$\sup_{\mathbf{D}_s(y)} w_2 = w_2(x_1) \leq w(x_1) = v(x_1) - \psi(x_1) + C_0 s^2 = u(x_1) - \psi(x_1) + C_0 s^2 \leq K s^2.$$

By combining these last two estimates we have proved (3.17).

Now let us pick $x \in \mathbf{D}_{s_0/2} \cap \{v > u\}$ and let y be the closest point to x in $\{u = v\}$. Since $u(0) = v(0)$, setting $s = |x - y|$ we have $s < s_0/2$. Moreover, considering that v is smooth inside $\mathbf{D}_r \cap \{u < v\}$, by (3.14) we find that

$$\sum_{i,j=1}^n a_{i,j} D_{ij}^2 v = \lambda \quad \text{on } \mathbf{D}_r \cap \{u < v\}$$

where

$$a = \frac{(1 + |\nabla v|^2) \text{Id} - \nabla v \otimes \nabla v}{(1 + |\nabla v|^2)^{3/2}}.$$

In particular,

$$\sum_{i,j=1}^n a_{i,j} D_{ij}^2 (v - u) = f \quad \text{on } \mathbf{D}_r \cap \{u < v\}$$

where

$$f = \lambda - \sum_{i,j=1}^n a_{i,j} D_{ij}^2 u.$$

Since $v \in C^{1,\beta}(\mathbf{D}_r)$, the matrix field a satisfies

$$\|a\|_{C^0(\mathbf{D}_r)} \leq N = N(n), \quad [a]_{C^{0,\beta}(\mathbf{D}_r)} \leq N = N(n, [\nabla v]_{C^{0,\beta}(\mathbf{D}_r)}) \quad (3.27)$$

and it is uniformly elliptic on \mathbf{D}_r , with ellipticity constant independent even from the dimension n thanks to $\text{Lip}(v) \leq 1$. Similarly, $f \in C^{0,\beta}(\mathbf{D}_r)$ with

$$\|f\|_{C^{0,\beta}(\mathbf{D}_r)} \leq N = N(n, \lambda, \|u\|_{C^{2,\beta}(\mathbf{D}_r)}, [\nabla v]_{C^{0,\beta}(\mathbf{D}_r)}). \quad (3.28)$$

By Schauder's theory for equations in non-divergence form applied on $\mathbf{D}_s(x) \subset \mathbf{D}_r \cap \{u < v\}$, see [GT98, Corollary 6.3] we find

$$\|\nabla^2(v - u)\|_{C^0(\mathbf{D}_{s/2}(x))} \leq N(n, \beta, r^\beta [a]_{C^{0,\beta}(\mathbf{D}_r(x))}) \left(\frac{\|v - u\|_{C^0(\mathbf{D}_s(x))}}{s^2} + s^\beta \|f\|_{C^{0,\beta}(\mathbf{D}_{2s}(x))} \right).$$

By combining (3.16), (3.17), (3.27) and (3.28) we find in particular that

$$|\nabla^2 v(x)| \leq C(n, \beta, \lambda, \|u\|_{C^{2,\beta}(\mathbf{D}_r)}, [\nabla v]_{C^{0,\beta}(\mathbf{D}_r)}) \quad \forall x \in \mathbf{D}_{s_0/2} \cap \{v > u\}.$$

Thus $v \in C^{1,1}(\mathbf{D}_{s_0/2}) = W^{2,\infty}(\mathbf{D}_{s_0/2})$, so that E_λ has $C^{1,1}$ -boundary in $\mathbf{C}_{s_0/2}$ and generalized bounded mean curvature in $\mathbf{C}_{s_0/2}$ satisfying

$$H_{E_\lambda}(x + v(x) e_{n+1}) = - \frac{(1 + |\nabla v(x)|^2) \Delta v(x) - \nabla^2 v(x) [\nabla v(x)] \cdot \nabla v(x)}{(1 + |\nabla v(x)|^2)^{3/2}} \quad (3.29)$$

for a.e. $x \in \mathbf{D}_{s_0/2}$ (thanks to (2.3)). Notice that $\nabla u = \nabla v$ on $\{u = v\} \cap \mathbf{D}_{s_0/2}$ and $\nabla^2 u(x) = \nabla^2 v(x)$ for a.e. $x \in \{u = v\} \cap \mathbf{D}_{s_0/2}$. In particular (3.29) gives us

$$H_{E_\lambda} = H_\Omega \quad \mathcal{H}^n\text{-a.e. on } \mathbf{C}_{s_0/2} \cap \partial E_\lambda \cap \partial \Omega$$

By a covering argument, and by taking into account that E_λ is smooth on $A_1 \cup A_2$ with $H_{E_\lambda} = H_\Omega$ on $A_1 \cap \partial E_\lambda$ and $H_{E_\lambda} = -\lambda$ on $A_2 \cap \partial E_\lambda$, we conclude that E_λ has $C^{1,1}$ -boundary in $\mathbb{R}^{n+1} \setminus \Sigma$, generalized bounded mean curvature in \mathbb{R}^{n+1} which satisfies $H_{E_\lambda}(x) \in \{H_\Omega(x), -\lambda\}$ for \mathcal{H}^n -a.e. $x \in \partial E_\lambda$. Recalling (3.9), we also have $H_{E_\lambda}(x) \geq -\lambda$ for \mathcal{H}^n -a.e. $x \in \partial E_\lambda$. This completes the proof of statement (iii), thus of the proposition. \square

Proof of Theorem 1.2. The first part of the statement, requiring the explicit assumption $\delta(\Omega) < P(B_1)$ only, was proved in Proposition 3.1. To complete the proof of the theorem, let us consider a sequence $\{\Omega_h\}_{h \in \mathbb{N}}$ open bounded sets in \mathbb{R}^{n+1} with smooth boundaries, such that $H_{\Omega_h} \leq n$ for every $h \in \mathbb{N}$, and

$$\delta(\Omega_h) = P(\Omega_h) - P(B_1) \rightarrow 0 \quad \text{as } h \rightarrow \infty.$$

Let $\{E_h\}_{h \in \mathbb{N}}$ be the sequence of sets associated to Ω_h by Proposition 3.2 with $\lambda = n$. In this way, each E_h is an open set with $C^{1,1}$ -boundary in $\mathbb{R}^{n+1} \setminus \Sigma_h$ (where Σ_h are closed sets with Hausdorff dimension at most $n-7$) and bounded generalized mean curvature in \mathbb{R}^{n+1} satisfying

$$\|H_{E_h}\|_{L^\infty(\partial E_h)} \leq \max \{ \|(H_{\Omega_h})^+\|_{C^0(\partial \Omega_h)}, n \} = n. \quad (3.30)$$

By the monotonicity formula for rectifiable sets with bounded generalized mean curvature [Sim83, Chapter 17], we have

$$\mathcal{H}^n(B_r(x) \cap \partial E_h) \geq c(n) r^n \quad \forall r < r(n),$$

so that, by a covering argument,

$$\text{diam}(E_h) \leq C(n) P(E_h).$$

Now, by Proposition 3.2, $\text{diam}(\Omega_h) = \text{diam}(E_h)$. At the same time, $\Omega_h \subset E_h$ and (3.2) imply

$$P(E_h) = \mathcal{H}^n(\partial E_h \cap \partial \Omega_h) + \mathcal{H}^n(\partial E_h \setminus \partial \Omega_h) \leq P(\Omega_h) + \delta(\Omega_h)$$

so that

$$\limsup_{h \rightarrow \infty} P(E_h) \leq P(B_1), \quad \text{diam}(E_h) = \text{diam}(\Omega_h) \leq C(n).$$

Let A_h be the convex hull of E_h . Up a translation $0 \in A_h$, so that $\nu_{A_h}(x) \cdot x \geq 0$ for every $x \in \partial A_h$, and in particular

$$\nu_{E_h}(x) \cdot x \geq 0 \quad \forall x \in \partial A_h \cap \partial E_h \setminus \Sigma_h. \quad (3.31)$$

Since $\text{diam}(E_h) \leq C(n)$ and $0 \in A_h$, we have $E_h \subset B_{C(n)}$ with $P(E_h) \leq P(B_1) + 1$ for every h large enough. By the standard compactness theorem for sets of finite perimeter, there exists a bounded set of finite perimeter E such that

$$|E_h \Delta E| \rightarrow 0 \quad \text{as } h \rightarrow \infty,$$

and hence, by lower semicontinuity of perimeter

$$P(E) \leq P(B_1). \quad (3.32)$$

We now exploit the divergence theorem

$$(n+1)|E_h| = \int_{\partial E_h} x \cdot \nu_{E_h} = \int_{\partial A_h \cap \partial E_h} x \cdot \nu_{E_h} + \int_{(\partial E_h) \setminus \partial A_h} x \cdot \nu_{E_h}$$

where

$$\left| \int_{(\partial E_h) \setminus \partial A_h} x \cdot \nu_{E_h} \right| \leq \text{diam}(E_h) \mathcal{H}^n(\partial E_h \setminus \partial A_h) \leq C(n) \delta(E_h).$$

where in the last step we have applied Almgren's identity (1.3) to E_h . Now, by (3.31) and since $n \geq H_{E_h} \geq 0$ on $\partial A_h \cap \partial E_h \setminus \Sigma_h$, for every $\varepsilon > 0$ we have

$$\int_{\partial A_h \cap \partial E_h} x \cdot \nu_{E_h} = \int_{\partial A_h \cap \partial E_h} \frac{H_{E_h} + \varepsilon}{H_{E_h} + \varepsilon} x \cdot \nu_{E_h} \geq \frac{1}{n + \varepsilon} \int_{\partial A_h \cap \partial E_h} (H_{E_h} + \varepsilon) (x \cdot \nu_{E_h}).$$

By (3.30),

$$\left| \int_{\partial E_h \setminus \partial A_h} (H_{E_h} + \varepsilon) (x \cdot \nu_{E_h}) \right| \leq (n + \varepsilon) \text{diam}(E_h) \mathcal{H}^n(\partial E_h \setminus \partial A_h) \leq C(n) \delta(E_h)$$

and thus, combining the above identities and estimates,

$$(n+1)|E_h| \geq \frac{1}{n + \varepsilon} \int_{\partial E_h} (H_{E_h} + \varepsilon) (x \cdot \nu_{E_h}) - C(n) \delta(E_h).$$

By the tangential divergence theorem

$$\int_{\partial E_h} (H_{E_h} + \varepsilon) (x \cdot \nu_{E_h}) = n P(E_h) + \varepsilon (n+1) |E_h|$$

so that

$$(n+1)|E_h| \geq \frac{n P(E_h) + \varepsilon (n+1) |E_h|}{n + \varepsilon} - C(n) \delta(E_h).$$

We let $\varepsilon \rightarrow 0$ and apply Almgren's principle $P(E_h) \geq P(B_1)$ to conclude that

$$(n+1)|E_h| \geq P(B_1) - C(n) \delta(E_h) = (n+1)|B_1| - C(n) \delta(E_h),$$

that is

$$|B_1| - |E_h| \leq C(n) \delta(E_h). \quad (3.33)$$

Let us now assume that $|E_h| > |B_1|$ and let $\lambda_h = (|B_1|/|E_h|)^{1/(n+1)}$ so that $|\lambda_h E_h| = |B_1|$ and thus $P(\lambda_h E_h) \geq P(B_1)$ by the isoperimetric inequality. By $1 - \lambda_h^n \geq 1 - \lambda_h$ we thus find

$$P(E_h) - P(B_1) = (1 - \lambda_h^n) P(E_h) + P(\lambda_h E_h) - P(B_1) \geq (1 - \lambda_h) P(E_h).$$

Since $P(E_h) \rightarrow P(B_1)$ we conclude that

$$C(n) \delta(E_h) \geq 1 - \lambda_h = 1 - \left(\frac{|B_1|}{|E_h|} \right)^{1/(n+1)}$$

that is $|E_h| - |B_1| \leq C(n) \delta(E_h)$. Also taking (3.33) into account we thus find

$$||E_h| - |B_1|| \leq C(n) \delta(E_h).$$

In particular, $|E| = |B_1|$ and thus (3.32) and the isoperimetric theorem imply that $E = B_1$ (up to a final translation). Since $\partial E = \mathbb{S}^n$ is a smooth hypersurface and $|E_h \Delta E| \rightarrow 0$ as $h \rightarrow \infty$ with $\|H_{E_h}\|_{L^\infty(\partial E_h)} \leq n$, by applying Allard's regularity theorem we find that $\Sigma_h = \emptyset$ and that there are maps $u_h : \mathbb{S}^n \rightarrow \mathbb{R}$ such that

$$\partial E_h = \{x + u_h(x) x : x \in \mathbb{S}^n\} \quad \lim_{h \rightarrow \infty} \|u_h\|_{C^1(\mathbb{S}^n)} = 0.$$

(Referring to [CM15, Lemma 2.8] or [CLM14, Lemma 4.4] for more details on this point, we just mention that here the idea is that of exploiting the continuity of area excess on a fixed cylinder along sequences of almost-minimizers. By choosing a scale such that the area excess of \mathbb{S}^n is suitably small with respect to the regularity threshold from Allard's theorem – notice that here we do not have to care about multiplicities as we are working with boundaries of finite perimeter sets – we deduce by continuity that the area excess of E_h on a cylinder of such scale is going to be below Allard's regularity threshold.) This concludes the proof of the theorem. \square

4. SHARP STABILITY ESTIMATES FOR C^1 -SMALL NORMAL DEFORMATIONS OF \mathbb{S}^n

This section is devoted to the proof of the estimates in Theorem 1.5, that is say, to the quantitative stability problem for C^1 -small normal deformations of the sphere. (The sharpness of these estimates is discussed in the next section.) We divide the proof into a series of lemmas, throughout which we shall always consider the following assumptions:

$$\Omega \text{ is an open set with } C^{1,1}\text{-boundary } \partial\Omega = \{(1 + u(x)) x : x \in \mathbb{S}^n\} \quad (4.1)$$

$$H_\Omega \leq n \text{ a.e. on } \partial\Omega \quad (4.2)$$

$$\|u\|_{C^1(\mathbb{S}^n)} \leq \varepsilon(n) \quad (4.3)$$

$$\int_{\partial\Omega} x = 0. \quad (4.4)$$

Notice that

$$\delta(\Omega) = P(\Omega) - P(B_1)$$

can be made arbitrarily small thanks to (4.1) and (4.3), and is non-negative by Almgren's principle.

Lemma 4.1. *If (4.1), (4.2), (4.3) and (4.4) hold, then*

$$\frac{\delta(\Omega)}{C(n)} \leq \int_{\mathbb{S}^n} u \leq C(n) \delta(\Omega), \quad (4.5)$$

$$\|u\|_{W^{1,2}(\mathbb{S}^n)}^2 \leq C(n) (\delta(\Omega)^2 + \delta(\Omega) \|u\|_{C^0(\mathbb{S}^n)}), \quad (4.6)$$

$$\|u\|_{C^0(\mathbb{S}^n)} \leq C(n) \begin{cases} \delta(\Omega), & \text{if } n = 1, \\ \delta(\Omega) \log\left(\frac{1}{\delta(\Omega)}\right), & \text{if } n = 2 \\ \delta(\Omega)^{1/(n-1)}, & \text{if } n \geq 3. \end{cases} \quad (4.7)$$

Proof of Lemma 4.1. Following the approach of Fuglede [Fug89], the proof consists in expanding u into spherical harmonics, and then obtaining the desired estimates by combining the Taylor expansions of $\delta(\Omega)$ and $\int_{\partial\Omega} x$ with the vanishing barycenter condition (4.4) and with the inequality obtained by testing the non-negative function $n - H_\Omega$ (computed in the coordinates of \mathbb{S}^n) against $u - \inf_{\mathbb{S}^n} u$. We notice that assumption (4.2) will not be used until step four of the proof.

Step one: We start by expressing u into spherical harmonics. Constants and coordinate functions provide the first $n + 2$ spherical harmonics on \mathbb{S}^n . Correspondingly we have

$$u(x) = a + b \cdot x + R(x) \quad x \in \mathbb{S}^n, \quad (4.8)$$

where

$$a = \frac{1}{\mathcal{H}^n(\mathbb{S}^n)} \int_{\mathbb{S}^n} u \quad b_i = \frac{\int_{\mathbb{S}^n} x_i u}{\int_{\mathbb{S}^n} x_i^2} \quad i = 1, \dots, n+1,$$

and $R \in C^1(\mathbb{S}^n)$ satisfies

$$\int_{\mathbb{S}^n} R = 0 \quad \int_{\mathbb{S}^n} x R = 0. \quad (4.9)$$

The following remarks will be useful. First, by $\int_{\mathbb{S}^n} (b \cdot x)^2 = |b|^2 \mathcal{H}^n(\mathbb{S}^n)/(n+1)$, we have

$$\int_{\mathbb{S}^n} u^2 = \int_{\mathbb{S}^n} a^2 + (b \cdot x)^2 + R^2 = \mathcal{H}^n(\mathbb{S}^n) \left(a^2 + \frac{|b|^2}{n+1} \right) + \int_{\mathbb{S}^n} R^2 \quad (4.10)$$

$$\int_{\mathbb{S}^n} |\nabla u|^2 = \int_{\mathbb{S}^n} |b - (b \cdot x)x|^2 + |\nabla R|^2. \quad (4.11)$$

In particular, by (4.10), (4.11) and

$$0 = \int_{\mathbb{S}^n} n(b \cdot x)^2 - |b - (b \cdot x)x|^2,$$

we find that

$$\begin{aligned} \int_{\mathbb{S}^n} n u^2 - |\nabla u|^2 &= \int_{\mathbb{S}^n} n a^2 + n(b \cdot x)^2 + n R^2 - |b - (b \cdot x)x|^2 - |\nabla R|^2 \\ &= \int_{\mathbb{S}^n} n a^2 + n R^2 - |\nabla R|^2. \end{aligned} \quad (4.12)$$

Second, since the k -th eigenvalue λ_k of the Laplacian over \mathbb{S}^n satisfies $\lambda_k = k(n+k-1)$, and since R is orthogonal to the first two eigenspaces (see (4.9)), for R we have the Poincaré-type inequality

$$\int_{\mathbb{S}^n} |\nabla R|^2 \geq 2(n+1) \int_{\mathbb{S}^n} R^2, \quad (4.13)$$

which is stronger than the usual Poincaré inequality

$$\int_{\mathbb{S}^n} |\nabla v|^2 \geq n \int_{\mathbb{S}^n} v^2 \quad \forall v \in W^{1,2}(\mathbb{S}^n) \text{ with } \int_{\mathbb{S}^n} v = 0.$$

Finally, by (4.3) we have

$$|a| \leq \varepsilon(n) \quad |b| \leq C(n) \varepsilon(n) \quad \|R\|_{C^1(\mathbb{S}^n)} \leq C(n) \varepsilon(n). \quad (4.14)$$

This fact will be particularly useful in expanding the metric of S as seen from \mathbb{S}^n ,

$$G(x) = (G_{ij}(x)) = ((1+u(x))^2 \delta_{ij} + \nabla_{\tau_i} u(x) \nabla_{\tau_j} u(x)) \quad x \in \mathbb{S}^n.$$

Here τ_1, \dots, τ_n is an orthonormal basis for $T_x \mathbb{S}^n$ at $x \in \mathbb{S}^n$, and using (4.3), we compute

$$\begin{aligned} G^{-1} &= (G^{ij}(x)) = \left(\frac{\delta_{ij}}{(1+u)^2} - \frac{\nabla_{\tau_i} u \nabla_{\tau_j} u}{(1+u)^2((1+u)^2 + |\nabla u|^2)} \right) \\ \det G &= (1+u)^{2n-2}((1+u)^2 + |\nabla u|^2), \\ \sqrt{\det G} &= (1+u)^{n-1} \sqrt{(1+u)^2 + |\nabla u|^2}. \end{aligned} \quad (4.15)$$

Step two: We now exploit the barycenter assumption (4.4) to show that

$$|b| \leq C(n) \int_{\mathbb{S}^n} |\nabla R|^2 + \varepsilon O(u^2 + |\nabla u|^2). \quad (4.16)$$

Indeed, by the area formula, (4.4) takes the form

$$0 = \int_{\partial\Omega} x = \int_{\mathbb{S}^n} (1+u) x \sqrt{\det G}. \quad (4.17)$$

By (4.15) and (4.3) we have

$$(1+u)\sqrt{\det G} = 1 + (n+1)u + (n+1)n\frac{u^2}{2} + \frac{|\nabla u|^2}{2} + \varepsilon O(u^2 + |\nabla u|^2),$$

where $O(u^2 + |\nabla u|^2)$ denotes a function of \mathbb{S}^n bounded in absolute value by $C(n)(u^2 + |\nabla u|^2)$. By $\int_{\mathbb{S}^n} x = 0$ and (4.9) we find

$$\int_{\mathbb{S}^n} x u = a \int_{\mathbb{S}^n} x + \int_{\mathbb{S}^n} (b \cdot x) x + \int_{\mathbb{S}^n} x R = \int_{\mathbb{S}^n} (b \cdot x) x = b \int_{\mathbb{S}^n} x_1^2$$

and similarly

$$\begin{aligned} \int_{\mathbb{S}^n} x u^2 &= a^2 \int_{\mathbb{S}^n} x + 2a \int_{\mathbb{S}^n} x(b \cdot x + R) + \int_{\mathbb{S}^n} x(b \cdot x + R)^2 \\ &= 2ab \int_{\mathbb{S}^n} x_1^2 + \int_{\mathbb{S}^n} x(b \cdot x + R)^2 \end{aligned}$$

By combining (4.17) with these identities we thus find

$$\mathcal{H}^n(\mathbb{S}^n)(1+na)b = -\frac{n(n+1)}{2} \int_{\mathbb{S}^n} x(b \cdot x + R)^2 - \int_{\mathbb{S}^n} x \frac{|\nabla u|^2}{2} + \varepsilon O(u^2 + |\nabla u|^2)$$

so that, by (4.14) and by noticing that $|\nabla u| \leq |b| + |\nabla R|$,

$$\begin{aligned} |b| &\leq C(n) \int_{\mathbb{S}^n} |b|^2 + R^2 + |\nabla R|^2 + \varepsilon O(u^2 + |\nabla u|^2) \\ &\leq C(n) \int_{\mathbb{S}^n} |b|^2 + |\nabla R|^2 + \varepsilon O(u^2 + |\nabla u|^2), \end{aligned}$$

where in the last inequality we have used (4.13) (here using the weaker version with n in place of $2(n+1)$ would have been fine as well).

Step three: Now we compute H_Ω in the coordinates of \mathbb{S}^n , see (4.20) below. By assumption $H_\Omega \in L^1_{\text{loc}}(\mathcal{H}^n \llcorner \partial\Omega)$ is such that

$$\int_{\partial\Omega} \nu_\Omega \cdot X H_\Omega d\mathcal{H}^n = \int_{\partial\Omega} \operatorname{div}^{\partial\Omega} X d\mathcal{H}^n \quad \forall X \in C_c^\infty(\mathbb{R}^{n+1}; \mathbb{R}^{n+1}). \quad (4.18)$$

Set

$$H^*(x) := H_\Omega((1+u(x))x) \quad \forall x \in \mathbb{S}^n.$$

We test (4.18) with $X \in C_c^\infty(\mathbb{R}^{n+1}; \mathbb{R}^{n+1})$ such that, for some fixed $\zeta \in C_c^\infty(\mathbb{R}^{n+1})$ and for every $x \in \mathbb{S}^n$, one has

$$X((1+u(x))x) = \frac{d}{dt}(1+u(x) + t\zeta(x))x \Big|_{t=0} = \zeta(x)x.$$

By the area formula, (4.18) writes as

$$\int_{\mathbb{S}^n} H^* \zeta \nu_\Omega \cdot x \sqrt{\det G} = \int_{\mathbb{S}^n} \sqrt{\det G} G^{ij} (\nabla_{\tau_j} u \nabla_{\tau_i} \zeta + (1+u) \zeta \delta_{ij}). \quad (4.19)$$

We expand the both sides of (4.19) by means of (4.15) in order to find that

$$\begin{aligned} &\int_{\mathbb{S}^n} H^* \zeta (1+u)^n \\ &= \int_{\mathbb{S}^n} (1+u)^{n-3} \sqrt{(1+u)^2 + |\nabla u|^2} \left(\delta_{ij} - \frac{\nabla_{\tau_i} u \nabla_{\tau_j} u}{(1+u)^2 + |\nabla u|^2} \right) (\nabla_{\tau_j} u \nabla_{\tau_i} \zeta + (1+u) \zeta \delta_{ij}) \end{aligned}$$

$$= \int_{\mathbb{S}^n} (1+u)^{n-3} \left(\frac{(1+u)^2 \nabla u \cdot \nabla \zeta}{\sqrt{(1+u)^2 + |\nabla u|^2}} - \frac{(1+u) |\nabla u|^2 \zeta}{\sqrt{(1+u)^2 + |\nabla u|^2}} + n(1+u) \sqrt{(1+u)^2 + |\nabla u|^2} \zeta \right).$$

Replacing ζ with $(1+u)^{-n} \zeta$,

$$\int_{\mathbb{S}^n} H^* \zeta = \int_{\mathbb{S}^n} \left(\frac{\nabla u \cdot \nabla \zeta}{(1+u) \sqrt{(1+u)^2 + |\nabla u|^2}} + \frac{n \zeta}{\sqrt{(1+u)^2 + |\nabla u|^2}} - \frac{|\nabla u|^2 \zeta}{(1+u)^2 \sqrt{(1+u)^2 + |\nabla u|^2}} \right).$$

Since $\zeta \in C_c^\infty(\mathbb{R}^{n+1})$ is arbitrary, we can write H^* in divergence form as

$$H^* = -\operatorname{div}_{\mathbb{S}^n} \left(\frac{\nabla u}{(1+u) \sqrt{(1+u)^2 + |\nabla u|^2}} \right) + \frac{n - \frac{|\nabla u|^2}{(1+u)^2}}{\sqrt{(1+u)^2 + |\nabla u|^2}}, \quad (4.20)$$

which is the formula needed in the sequel.

Step four: We conclude the proof. By (4.3), (4.15), (4.10) and (4.11)

$$\begin{aligned} \delta(\Omega) &= \mathcal{H}^n(S) - \mathcal{H}^n(\mathbb{S}^n) \\ &= \int_{\mathbb{S}^n} (\sqrt{\det G} - 1) = \int_{\mathbb{S}^n} ((1+u)^{n-1} \sqrt{(1+u)^2 + |\nabla u|^2} - 1) \\ &= \int_{\mathbb{S}^n} \left(n u + \frac{n(n-1)}{2} u^2 + \frac{1}{2} |\nabla u|^2 \right) + \varepsilon O(\|u\|_{W^{1,2}(\mathbb{S}^n)}^2) \\ &= \mathcal{H}^n(\mathbb{S}^n) \left(n a + \frac{n(n-1)}{2} a^2 + \frac{n(n-1)}{2(n+1)} |b|^2 \right) + \frac{1}{2} \int_{\mathbb{S}^n} |b - (b \cdot x)x|^2 \\ &\quad + \int_{\mathbb{S}^n} \left(\frac{n(n-1)}{2} R^2 + \frac{|\nabla R|^2}{2} \right) + \varepsilon O(\|u\|_{W^{1,2}(\mathbb{S}^n)}^2), \end{aligned}$$

which thanks to (4.16) gives

$$\delta(\Omega) = \mathcal{H}^n(\mathbb{S}^n) \left(n a + \frac{n(n-1)}{2} a^2 \right) + \int_{\mathbb{S}^n} \left(\frac{n(n-1)}{2} R^2 + \frac{|\nabla R|^2}{2} \right) + \varepsilon O(\|u\|_{W^{1,2}(\mathbb{S}^n)}^2) \quad (4.21)$$

Now let $\ell = \sup_{\mathbb{S}^n} |u^-|$, where $u^-(x) = \max\{-u(x), 0\}$, so that $u + \ell \geq 0$ and $\ell \leq \varepsilon$ by (4.3). By (4.2), (4.20) and $\|u\|_{C^1(\mathbb{S}^n)} \leq \varepsilon$

$$\begin{aligned} 0 &\leq \int_{\mathbb{S}^n} (n - H^*) (u + \ell) \\ &= \int_{\mathbb{S}^n} \left(-\frac{|\nabla u|^2}{(1+u) \sqrt{(1+u)^2 + |\nabla u|^2}} + n(u + \ell) - \frac{n(u + \ell)}{\sqrt{(1+u)^2 + |\nabla u|^2}} \right. \\ &\quad \left. + \frac{(u + \ell) |\nabla u|^2}{(1+u)^2 \sqrt{(1+u)^2 + |\nabla u|^2}} \right) \\ &\leq \int_{\mathbb{S}^n} (n \ell u + n u^2 - |\nabla u|^2) + \varepsilon O(\|u\|_{W^{1,2}(\mathbb{S}^n)}^2). \end{aligned}$$

By combining this inequality with (4.12), we find that

$$\int_{\mathbb{S}^n} |\nabla R|^2 \leq n \ell a + \int_{\mathbb{S}^n} n a^2 + n R^2 + \varepsilon O(\|u\|_{W^{1,2}(\mathbb{S}^n)}^2).$$

By (4.13) (where now it is crucial to have $2(n+1)$ in place of n in the Poincaré inequality)

$$\left(1 - \frac{n}{2(n+1)} \right) \int_{\mathbb{S}^n} |\nabla R|^2 \leq n \ell a + n \mathcal{H}^n(\mathbb{S}^n) a^2 + \varepsilon O(\|u\|_{W^{1,2}(\mathbb{S}^n)}^2). \quad (4.22)$$

By combining (4.10), (4.11), (4.16) and (4.22) we find that

$$\int_{\mathbb{S}^n} u^2 + |\nabla u|^2 \leq C(n) (\ell a + a^2) \leq C(n) \varepsilon |a|. \quad (4.23)$$

Hence (4.21) gives $\delta(\Omega) = n\mathcal{H}^n(\mathbb{S}^n)a + \varepsilon O(|a|)$, that is

$$\frac{\delta(\Omega)}{C(n)} \leq a \leq C(n) \delta(\Omega). \quad (4.24)$$

This proves (4.5), and then the first inequality in (4.23) implies (4.6).

Step five: We finally prove (4.7). To this end, let us recall the following Poincaré-type interpolation inequality from [Fug89, Lemma 1.4]: *for every $v \in C^1(\mathbb{S}^n)$ with $\int_{\mathbb{S}^n} v = 0$, one has*

$$\|v\|_{C^0(\mathbb{S}^n)} \leq C(n) \begin{cases} \|\nabla v\|_{L^2(\mathbb{S}^1)} & \text{if } n = 1 \\ \|\nabla v\|_{L^2(\mathbb{S}^2)} \log \left(C(2) \frac{\|\nabla v\|_{C^0(\mathbb{S}^2)}}{\|\nabla v\|_{L^2(\mathbb{S}^2)}} \right)^{\frac{1}{2}} & \text{if } n = 2 \\ \|\nabla v\|_{C^0(\mathbb{S}^n)}^{\frac{n-2}{n}} \|\nabla v\|_{L^2(\mathbb{S}^n)}^{\frac{2}{n}} & \text{if } n > 2. \end{cases} \quad (4.25)$$

We deduce (4.7) by applying (4.25) to $v = u - a$. For example, in the case $n > 2$, by (4.5), (4.6) and (4.25) we find

$$\|u\|_{C^0(\mathbb{S}^n)} \leq a + \|v\|_{C^0(\mathbb{S}^n)} \leq C(n) \delta(\Omega) + C(n) \left(\delta(\Omega)^2 + \|u\|_{C^0(\mathbb{S}^n)} \delta(\Omega) \right)^{1/n}.$$

Assuming without loss of generality that $\delta(\Omega) \leq \|u\|_{C^0(\mathbb{S}^n)}/M(n)$ for a suitably large constant $M(n)$, we deduce that

$$\|u\|_{C^0(\mathbb{S}^n)} \leq C(n) \left(\delta(\Omega)^2 + \|u\|_{C^0(\mathbb{S}^n)} \delta(\Omega) \right)^{1/n} \leq C(n) \|u\|_{C^0(\mathbb{S}^n)}^{1/n} \delta(\Omega)^{1/n}$$

and thus $\|u\|_{C^0(\mathbb{S}^n)} \leq \delta(\Omega)^{1/(n-1)}$, as desired. The cases $n = 2$ and $n = 1$ follow by analogous arguments. This completes the proof of Lemma 4.1. \square

Taking into account Lemma 4.1, in order to complete the proof of Theorem 1.5 we are left to obtain linear bounds on $\|u\|_{L^1(\mathbb{S}^n)}$ and $\|u^+\|_{C^0(\mathbb{S}^n)}$.

Lemma 4.2. *If u and Ω satisfy (4.1), (4.3) and (4.4), then for every $q > n/2$*

$$\|u\|_{C^0(\mathbb{S}^n)} \leq C(n, q) \left(\|u\|_{L^2(\mathbb{S}^n)} + \|H_\Omega - n\|_{L^q(\partial\Omega)} \right), \quad (4.26)$$

and whenever $1 \leq p < n/(n-1)$,

$$\|\nabla u\|_{L^p(\mathbb{S}^n)} \leq C(n, p) \left(\|u\|_{L^p(\mathbb{S}^n)} + \|H_\Omega - n\|_{L^1(\partial\Omega)} \right). \quad (4.27)$$

In addition: (i) if $H_\Omega \leq n$ \mathcal{H}^n -a.e. on $\partial\Omega$, then

$$\|u^+\|_{C^0(\mathbb{S}^n)} \leq C(n) \|u\|_{L^1(\mathbb{S}^n)}; \quad (4.28)$$

(ii) if $p \in (1, \infty)$ and there exists $K > 0$ such that

$$\|\nabla u\|_{L^p(\mathbb{S}^n)} \leq K \|u\|_{L^p(\mathbb{S}^n)}, \quad (4.29)$$

then, provided $\|u\|_{C^1(\mathbb{S}^n)} \leq \varepsilon(n, p, K)$, one has

$$\|u\|_{L^p(\mathbb{S}^n)} \leq C(n, p, K) \left(\left| \int_{\mathbb{S}^n} u x \right| + \|H_\Omega - n\|_{L^1(\partial\Omega)} \right). \quad (4.30)$$

(iii) if $\alpha \in (0, 1)$ and there exists $K > 0$ such that $\|\nabla u\|_{C^{0,\alpha}(\mathbb{S}^n)} \leq K$, then

$$\|u\|_{C^{1,\alpha}(\mathbb{S}^n)} \leq C(n, K, \alpha) \left(\|u\|_{C^0(\mathbb{S}^n)} + \|H_\Omega - n\|_{L^\infty(\partial\Omega)} \right). \quad (4.31)$$

(iv) finally, if $\Lambda \geq 0$, $1 \leq p < n/(n-1)$, and

$$-\Lambda \leq H_\Omega \leq n \quad \mathcal{H}^n\text{-a.e. on } \partial\Omega \quad (4.32)$$

$$\|u\|_{C^1(\mathbb{S}^n)} \leq \varepsilon(n, p, \Lambda), \quad (4.33)$$

then

$$\|u\|_{W^{1,p}(\mathbb{S}^n)} \leq C(n, p, \Lambda) \delta(\Omega), \quad (4.34)$$

$$\|u^+\|_{C^0(\mathbb{S}^n)} \leq C(n, \Lambda) \delta(\Omega). \quad (4.35)$$

The proof is based on combining Almgren's identity (1.3) with two estimates from elliptic regularity theory, which are proved in Lemma 4.3 and Lemma 4.4 below.

Lemma 4.3. *If $n \geq 2$, $p \in (1, n/(n-1))$, $\rho > 0$, $f \in L^p(B_\rho; \mathbb{R}^n)$, $g \in L^1(B_\rho)$, and $v \in C^1(B_\rho)$ is a weak solution of*

$$\Delta v = \operatorname{div}(f) + g \quad \text{in } B_\rho, \quad (4.36)$$

then

$$\|\nabla v\|_{L^p(B_{\rho/2})} \leq C(n, p) \left(\rho^{-1} \|v\|_{L^p(B_\rho)} + \|f\|_{L^p(B_\rho)} + \rho^{1+\frac{n}{p}-n} \|g\|_{L^1(B_\rho)} \right). \quad (4.37)$$

Proof of Lemma 4.3. The argument is based on the use of standard elliptic estimates, and it is detailed just for the sake of clarity. (In particular, we could not find an exact reference for the case considered in here, where we need to use the L^1 -norm of g ; see (4.58) below.) By scaling we can set $\rho = 1$, and then prove (4.37) in three steps.

Step one: We assume that $f \in C_c^\infty(B_1; \mathbb{R}^n)$ and $g \in C_c^\infty(B_1)$. Denoting by Γ the fundamental solution of the Laplacian on \mathbb{R}^n , let us set $v^i = \Gamma \star (D_i f^i)$ and $w = \Gamma \star g$, where \star denotes convolution. In this way $\Delta v^i = D_i f^i$ and $\Delta w = g$ on \mathbb{R}^n (in pointwise sense), and defining φ by the identity

$$v = \varphi + \sum_{i=1}^n v^i + w$$

we have that φ is harmonic on B_1 , and thus such that

$$\|\nabla \varphi\|_{L^p(B_{1/2})} \leq C(n, p) \|\varphi\|_{L^p(B_1)}. \quad (4.38)$$

By the Calderon-Zygmund theory, $\Delta v^i = D_i f^i$ implies

$$\|\nabla v^i\|_{L^p(B_{1/2})} \leq C(n, p) \left(\|v^i\|_{L^p(B_1)} + \|f^i\|_{L^p(B_1)} \right). \quad (4.39)$$

Since $|\nabla \Gamma(z)| = c(n) |z|^{1-n}$ for every $n \geq 2$, thanks to $p < n/(n-1)$ we have

$$\begin{aligned} \int_{B_{1/2}} |Dw(x)|^p dx &\leq \int_{B_{1/2}} \left(\int_{\mathbb{R}^n} |x-y|^{1-n} |g(y)| dy \right)^p dx \\ &\leq \int_{B_{1/2}} \left(\int_{\mathbb{R}^n} |x-y|^{p(1-n)} |g(y)| dy \right) \left(\int_{\mathbb{R}^n} |g(y)| dy \right)^{p-1} dx \\ &= \|g\|_{L^1(B_1)}^{p-1} \int_{\mathbb{R}^n} |g(y)| dy \int_{B_{1/2}} |x-y|^{p(1-n)} dx \\ &\leq C(n, p) \|g\|_{L^1(B_1)}^p. \end{aligned} \quad (4.40)$$

By combining (4.38), (4.39) and (4.40) we obtain (4.37) when $f \in C_c^\infty(B_1; \mathbb{R}^n)$ and $g \in C_c^\infty(B_1)$.

Step two: Now we assume that $f \in L^p(B_1; \mathbb{R}^n)$, $g \in L^1(B_1)$ with $f = 0$ and $g = 0$ on $\mathbb{R}^n \setminus B_{3/4}$. Let us fix an even function $\rho \in C_c^\infty(B_1)$ with $0 \leq \rho \leq 1$ and $\int_{\mathbb{R}^n} \rho = 1$, set $\rho_\delta(z) = \delta^{-n} \rho(z/\delta)$ for $z \in \mathbb{R}^n$, and define $f_\delta = f \star \rho_\delta$, $g_\delta = g \star \rho_\delta$ and $v_\delta = \bar{v} \star \rho_\delta$, where \bar{v} is the extension to zero

of v outside of B_1 . If $\delta < 1/4$, then $f_\delta \in C_c^\infty(B_1; \mathbb{R}^n)$, $g_\delta \in C_c^\infty(B_1)$ and v_δ is a weak solution in $B_{3/4}$ of $\Delta v_\delta = \operatorname{div}(f_\delta) + g_\delta$. By case one

$$\|\nabla v_\delta\|_{L^p(B_{1/2})} \leq C(n, p) \left(\|v_\delta\|_{L^p(B_{3/4})} + \|f_\delta\|_{L^p(B_{3/4})} + \|g_\delta\|_{L^1(B_{3/4})} \right). \quad (4.41)$$

Since $v_\delta \rightarrow v$ in $W^{1,p}(B_{3/4})$, $f_\delta \rightarrow f$ in $L^p(B_{3/4})$ and $g_\delta \rightarrow g$ in $L^1(B_{3/4})$, letting $\delta \rightarrow 0^+$ in (4.41) we deduce (4.37) in this case too.

Step three: We finally prove (4.37) in full generality. Let $\eta \in C_c^\infty(B_{3/4})$ with $0 \leq \eta \leq 1$ and $\eta = 1$ on $B_{1/2}$. If $w = \eta v$, then, in distributional sense,

$$\begin{aligned} \Delta w &= \eta (\operatorname{div} f + g) + 2\nabla \eta \cdot \nabla v + v \Delta \eta \\ &= \operatorname{div}(\eta f + 2v \nabla \eta) - f \cdot \nabla \eta - v \Delta \eta + \eta g \\ &= \operatorname{div}(\bar{f}) + \bar{g} \quad \text{on } B_1 \end{aligned}$$

provided $\bar{f} = f \eta + 2v \nabla \eta \in L^p(B_1)$ and $\bar{g} = -f \cdot \nabla \eta - v \Delta \eta + \eta g$. Since \bar{f} and \bar{g} vanish outside $B_{3/4}$, by step two we can apply (4.37) to w and exploit $\nabla v = \nabla w$ on $B_{1/2}$ together with

$$\|w\|_{L^p(B_1)} + \|\bar{f}\|_{L^p(B_1)} + \|\bar{g}\|_{L^1(B_1)} \leq C(n, p) \left(\|v\|_{L^p(B_1)} + \|f\|_{L^p(B_1)} + \|g\|_{L^1(B_1)} \right),$$

to complete the proof of (4.37) in the general case. \square

Lemma 4.4. *For every $n \geq 2$, $K > 0$ and $p \in (1, \infty)$ there exist positive constants $C(n, p, K)$ and $\varepsilon(n, p, K)$ with the following property. If $G \in L^\infty(\mathbb{S}^n)$, $\alpha, \gamma \in C^0(\mathbb{S}^n)$ and $\beta \in C^0(\mathbb{S}^n, T\mathbb{S}^n)$ are such that*

$$\max \left\{ \|\alpha - 1\|_{C^0(\mathbb{S}^n)}, \|\beta\|_{C^0(\mathbb{S}^n)}, \|\gamma - n\|_{C^0(\mathbb{S}^n)} \right\} \leq \varepsilon(n, p, K) \quad (4.42)$$

and if $u \in C^1(\mathbb{S}^n)$ is a weak solution to

$$\operatorname{div}_{\mathbb{S}^n}(\alpha \nabla u) + \beta \cdot \nabla u + \gamma u = G \quad \text{on } \mathbb{S}^n \quad (4.43)$$

such that

$$\|\nabla u\|_{L^p(\mathbb{S}^n)} \leq K \|u\|_{L^p(\mathbb{S}^n)}, \quad (4.44)$$

then

$$\|u\|_{L^p(\mathbb{S}^n)} \leq C(n, p, K) \left(\left| \int_{\mathbb{S}^n} u x \right| + \|G\|_{L^1(\mathbb{S}^n)} \right). \quad (4.45)$$

Proof of Lemma 4.4. We argue by contradiction and assume the existence, for every $k \in \mathbb{N}$, of $G_k \in L^\infty(\mathbb{S}^n)$, $\alpha_k, \gamma_k \in C^0(\mathbb{S}^n)$, $\beta_k \in C^0(\mathbb{S}^n; T\mathbb{S}^n)$, and $u_k \in C^1(\mathbb{S}^n)$ such that

$$\lim_{k \rightarrow \infty} \max \left\{ \|\alpha_k - 1\|_{C^0(\mathbb{S}^n)}, \|\beta_k\|_{C^0(\mathbb{S}^n)}, \|\gamma_k - n\|_{C^0(\mathbb{S}^n)} \right\} = 0 \quad (4.46)$$

$$\operatorname{div}_{\mathbb{S}^n}(\alpha_k \nabla u_k) + \beta_k \cdot \nabla u_k + \gamma_k u_k = G_k \quad \text{weakly on } \mathbb{S}^n$$

with $\|\nabla u_k\|_{L^p(\mathbb{S}^n)} \leq K \|u_k\|_{L^p(\mathbb{S}^n)}$ for every $k \in \mathbb{N}$, and

$$\frac{\|u_k\|_{L^p(\mathbb{S}^n)}}{k} \geq \left| \int_{\mathbb{S}^n} u_k x \right| + \|G_k\|_{L^1(\mathbb{S}^n)}. \quad (4.47)$$

If we set $\bar{u}_k = \|u_k\|_{L^p(\mathbb{S}^n)}^{-1} u_k$ and $\bar{G}_k = \|u_k\|_{L^p(\mathbb{S}^n)} G_k$, then $\|\bar{G}_k\|_{L^1(\mathbb{S}^n)} \rightarrow 0$ and $\left| \int_{\mathbb{S}^n} \bar{u}_k x \right| \rightarrow 0$ as $k \rightarrow \infty$ with

$$\operatorname{div}_{\mathbb{S}^n}(\alpha_k \nabla \bar{u}_k) + \beta_k \cdot \nabla \bar{u}_k + \gamma_k \bar{u}_k = \bar{G}_k \quad \text{weakly on } \mathbb{S}^n \quad (4.48)$$

and $\|\nabla \bar{u}_k\|_{L^p(\mathbb{S}^n)} \leq K$ for every $k \in \mathbb{N}$. Since $p \in (1, \infty)$, there exists $u \in W^{1,p}(\mathbb{S}^n)$ such that $\bar{u}_k \rightarrow u$ in $L^p(\mathbb{S}^n)$ (so that $\|u\|_{L^p(\mathbb{S}^n)} = 1$ and $\int_{\mathbb{S}^n} u x = 0$) and $\nabla \bar{u}_k \rightarrow \nabla u$ in $L^p(\mathbb{S}^n)$. By (4.46) and since $\|\bar{G}_k\|_{L^1(\mathbb{S}^n)} \rightarrow 0$ as $k \rightarrow \infty$, we deduce from (4.48) that

$$\Delta u + n u = 0 \quad \text{on } \mathbb{S}^n.$$

Since $\|u\|_{L^p(\mathbb{S}^n)} = 1$, u is an eigenvector of the Laplacian on \mathbb{S}^n corresponding to the eigenvalue $\lambda_1 = n$. In particular, $u = c(n, p) x \cdot e$ for some unit vector e . This contradicts $\int_{\mathbb{S}^n} u x = 0$ and thus completes the proof of the lemma. \square

Proof of Lemma 4.2. Fix $e \in \mathbb{S}^n$, and set

$$\begin{aligned} \mathbf{K}_r &= \{x \in \mathbb{R}^{n+1} : |x - (x \cdot e)e| < r, x \cdot e > 0\}, \\ \mathbf{D}_r &= \{z \in e^\perp : |z| < r\} \end{aligned} \quad (4.49)$$

so that $\mathbf{K}_r = \mathbf{D}_r \times \mathbb{R}_+ e$. If $w_0(z) = \sqrt{1 - |z|^2}$, then

$$\mathbb{S}^n \cap \mathbf{K}_1 = \{x \in \mathbf{K}_1 : x \cdot e = w_0(x - (x \cdot e)e)\},$$

and

$$-\operatorname{div} \left(\frac{\nabla w_0}{\sqrt{1 + |\nabla w_0|^2}} \right) = n \quad \text{on } \mathbf{D}_1. \quad (4.50)$$

Thanks to $\|u\|_{C^1(\mathbb{S}^n)} \leq \varepsilon(n)$, we can find $w \in C^{1,1}(\mathbf{D}_{1/2})$ with $\operatorname{Lip}(w) \leq C(n)$ such that

$$\partial\Omega \cap \mathbf{K}_{1/2} = \{x \in \mathbf{K}_{1/2} : x \cdot e = w(x - (x \cdot e)e)\}.$$

Let us define $h \in L^\infty(\mathbf{D}_{1/2})$ by setting, for a.e. $z \in \mathbf{D}_{1/2}$,

$$h(z) = H_\Omega(z + w(z)e) = -\operatorname{div} \left(\frac{\nabla w(z)}{\sqrt{1 + |\nabla w(z)|^2}} \right).$$

Setting for $z \in \mathbf{D}_{1/2}$ and $\xi \in \mathbb{R}^n$

$$v(z) = w(z) - w_0(z), \quad F(\xi) = \frac{\xi}{\sqrt{1 + |\xi|^2}}, \quad M(z) = \int_0^1 \nabla F(\nabla w_0(z) + t \nabla v(z)) dt \quad (4.51)$$

we find that $F(\nabla w) - F(\nabla w_0) = M \nabla v$ and thus

$$h - n = -\operatorname{div} (F(\nabla w) - F(\nabla w_0)) = -\operatorname{div} (M \nabla v) = -\Delta v - \operatorname{div} ((M - \operatorname{Id}) \nabla v) \quad (4.52)$$

holds on $\mathbf{D}_{1/2}$. We now argue as follows:

Proof of (4.26): By the De Giorgi-Nash-Moser theorem (see, e.g. [GT98, Theorem 8.17]), since $\operatorname{div} (M \nabla v) = n - h$ on $\mathbf{D}_{1/2}$, we find

$$\|v\|_{C^0(\mathbf{D}_{1/4})} \leq C(n, q) \left(\|v\|_{L^2(\mathbf{D}_{1/2})} + \|n - h\|_{L^q(\mathbf{D}_{1/2})} \right) \quad \forall q > \frac{n}{2},$$

which immediately implies (4.26) thanks to a covering argument.

Proof of (4.27): If we set

$$g = n - h \quad f = (\operatorname{Id} - M) \nabla v,$$

then $g \in L^\infty(\mathbf{D}_{1/2})$ and $f \in C^0(\mathbf{D}_{1/2}; \mathbb{R}^n)$ with

$$\|g\|_{L^1(\mathbf{D}_{1/2})} \leq C(n) \|H - n\|_{L^1(\partial\Omega)} \quad \|f\|_{L^p(\mathbf{D}_\rho)} \leq C(n) (\varepsilon + \rho) \|\nabla v\|_{L^p(\mathbf{D}_\rho)}.$$

for every $\rho \in (0, 1/2)$, where we have used $\|u\|_{C^1(\mathbb{S}^n)} \leq \varepsilon$ and $\nabla w_0(0) = 0$ to deduce

$$\|M - \operatorname{Id}\|_{C^0(\mathbf{D}_\rho)} \leq C(\|\nabla w_0\|_{C^0(\mathbf{D}_\rho)} + \|\nabla v\|_{C^0(\mathbf{D}_\rho)}) \leq C(n) (\varepsilon + \rho).$$

Since v solves $\Delta v = \operatorname{div} (f) + g$ in $\mathbf{D}_{1/2}$, by Lemma 4.3 we find that, for every $\rho \in (0, 1/2)$,

$$\begin{aligned} \|\nabla v\|_{L^p(\mathbf{D}_{\rho/2})} &\leq C(n, p) \left(\rho^{-1} \|v\|_{L^p(\mathbf{D}_\rho)} + \|f\|_{L^p(\mathbf{D}_\rho)} + \rho^{1+\frac{n}{p}-n} \|g\|_{L^1(\mathbf{D}_\rho)} \right) \\ &\leq C(n, p) \left(\rho^{-1} \|v\|_{L^p(\mathbf{D}_\rho)} + (\varepsilon + \rho) \|\nabla v\|_{L^p(\mathbf{D}_\rho)} + \rho^{1+\frac{n}{p}-n} \|H - n\|_{L^1(\partial\Omega)} \right). \end{aligned}$$

If we denote by $G_r(e)$ the geodesic ball on \mathbb{S}^n of radius $r > 0$ and center $e \in \mathbb{S}^n$, then this last estimate implies, in terms of u , that, for every $\rho \in (0, 1/4)$

$$\|\nabla u\|_{L^p(G_\rho(e))} \leq C_0(n, p) \left(\rho^{-1} \|u\|_{L^p(G_{4\rho}(e))} + (\varepsilon + \rho) \|\nabla u\|_{L^p(G_{4\rho}(e))} + \rho^{1+\frac{n}{p}-n} \|H - n\|_{L^1(\partial\Omega)} \right). \quad (4.53)$$

Let us set, for $\rho_0 = \rho_0(n, p) \in (0, 1/4)$ to be determined,

$$Q = \sup \{ \|\nabla u\|_{L^p(G_\rho(e))} : \rho \in (\rho_0/2, \rho_0), e \in \mathbb{S}^n \}.$$

Clearly there exists $N(n)$ such that for every $e \in \mathbb{S}^n$ and $\rho < 1/4$ one can find $\{e_k\}_{k=1}^{N(n)} \subset \mathbb{S}^n$ such that

$$G_\rho(e) \subset \bigcup_{k=1}^{N(n)} G_{\rho/4}(e_k).$$

In this way by (4.53) and definition of Q we find that if $\rho \in (\rho_0/2, \rho_0)$, then

$$\begin{aligned} \|\nabla u\|_{L^p(G_\rho(e))}^p &\leq \sum_{k=1}^N \|\nabla u\|_{L^p(G_{\rho/4}(e_k))}^p \\ &\leq C_0(n, p)^p \sum_{k=1}^{N(n)} \left(\frac{\|u\|_{L^p(\mathbb{S}^n)}}{\rho} + (\varepsilon + \rho_0) \|\nabla u\|_{L^p(G_{\rho}(e_k))} + \rho^{1+\frac{n}{p}-n} \|H - n\|_{L^1(\partial\Omega)} \right)^p \\ &\leq C_0(n, p)^p N(n) \left(\frac{2\|u\|_{L^p(\mathbb{S}^n)}}{\rho_0} + (\varepsilon + \rho_0) Q + \rho_0^{1+\frac{n}{p}-n} \|H - n\|_{L^1(\partial\Omega)} \right)^p, \end{aligned}$$

that is

$$Q \leq C_0(n, p) N(n)^{1/p} \left(\frac{2\|u\|_{L^p(\mathbb{S}^n)}}{\rho_0} + (\varepsilon + \rho_0) Q + \rho_0^{1+\frac{n}{p}-n} \|H - n\|_{L^1(\partial\Omega)} \right).$$

Provided ε and ρ_0 are small enough in terms of n and p , we conclude that

$$\|\nabla u\|_{L^p(G_\rho(e))} \leq C(n, p) (\|u\|_{L^p(\mathbb{S}^n)} + \|H - n\|_{L^1(\partial\Omega)}), \quad \forall \rho \in (\rho_0/2, \rho_0), e \in \mathbb{S}^n,$$

so that, by a covering argument, we obtain (4.27).

Proof of (4.28): Recall that for proving (4.28) we are assuming $H_\Omega \leq n$ \mathcal{H}^n -a.e. on $\partial\Omega$, so that, by definition of h , we have $h(z) \leq n$ a.e. on $\mathbf{D}_{1/2}$. Coming back to (4.52) we thus see that $v = w - w_0$ solves

$$\operatorname{div}(M \nabla v) = n - h \geq 0 \quad \text{on } \mathbf{D}_{1/2}$$

that is, v is a subsolution to a quasilinear elliptic equation on $\mathbf{D}_{1/2}$. By Moser's iteration technique we find that

$$\|v^+\|_{C^0(\mathbf{D}_{1/4})} \leq C(n) \|v\|_{L^1(\mathbf{D}_{1/2})},$$

and thanks to the arbitrariness of e , we conclude the proof of (4.28).

Proof of (4.31): Since we are assuming that $\|\nabla u\|_{C^{0,\alpha}(\mathbb{S}^n)} \leq K$, we have $\|\nabla v\|_{C^{0,\alpha}(\mathbf{D}_{1/2})} \leq C(n, K)$, and thus looking back at the definition (4.51) of M , that

$$\|M\|_{C^{0,\alpha}(\mathbf{D}_{1/2})} \leq C(n, K).$$

We can thus apply [GT98, Theorem 8.32] to find that

$$\|v\|_{C^{1,\alpha}(\mathbf{D}_{1/4})} \leq C(n, K, \alpha) \left(\|v\|_{C^0(\mathbf{D}_{1/2})} + \|g\|_{L^\infty(\mathbf{D}_{1/2})} \right),$$

and then deduce (4.31) by a covering argument.

Proof of (4.30): Let us recall (4.20), namely

$$H^* = -\operatorname{div}_{\mathbb{S}^n} \left(\frac{\nabla u}{(1+u) \sqrt{(1+u)^2 + |\nabla u|^2}} \right) + \frac{n - \frac{|\nabla u|^2}{(1+u)^2}}{\sqrt{(1+u)^2 + |\nabla u|^2}}. \quad (4.54)$$

If we set

$$\alpha = \frac{1}{(1+u) \sqrt{(1+u)^2 + |\nabla u|^2}} \quad \gamma = \frac{n}{u} \left(1 - \frac{1}{1+u} \right) \quad G = H^* - n,$$

and define $\beta : \mathbb{S}^n \rightarrow T\mathbb{S}^n$ so that

$$\frac{|\nabla u|^2}{(1+u)^2 \sqrt{(1+u)^2 + |\nabla u|^2}} + n \left(\frac{1}{1+u} - \frac{1}{\sqrt{(1+u)^2 + |\nabla u|^2}} \right) = \beta \cdot \nabla u,$$

then (4.54) takes the form

$$\operatorname{div}_{\mathbb{S}^n} (\alpha \nabla u) + \beta \cdot \nabla u + \gamma u = G \quad \text{on } \mathbb{S}^n \quad (4.55)$$

where $\alpha \in C^0(\mathbb{S}^n)$, $\beta \in C^0(\mathbb{S}^n; T\mathbb{S}^n)$, $\gamma \in C^0(\mathbb{S}^n)$ and $G \in L^\infty(\mathbb{S}^n)$ are such that

$$\max \{ \|\alpha - 1\|_{C^0(\mathbb{S}^n)}, \|\beta\|_{C^0(\mathbb{S}^n)}, \|\gamma - n\|_{C^0(\mathbb{S}^n)} \} \leq C(n) \|u\|_{C^1(\mathbb{S}^n)} \quad (4.56)$$

$$\|G\|_{L^1(\mathbb{S}^n)} \leq C(n) \|H - n\|_{L^1(\partial\Omega)} \quad \|G\|_{L^\infty(\mathbb{S}^n)} = \|H - n\|_{L^\infty(\partial\Omega)}. \quad (4.57)$$

Thus, given $K > 0$ such that (4.29) holds, the validity of (4.30) follows by assuming $\|u\|_{C^1(\mathbb{S}^n)} \leq \varepsilon(n, p, K)$ and thanks to Lemma 4.4.

Conclusion of the proof: We finally assume the validity of (4.32) and (4.33) and prove (4.34) and (4.35). We first notice that by (1.3), denoting by A the convex envelope of Ω , we have

$$\delta(\Omega) \geq \mathcal{H}^n(\partial\Omega \setminus \partial A) + \int_{\partial A \cap \partial\Omega} \left(1 - \left(\frac{H_\Omega}{n} \right)^n \right).$$

Since $0 \leq H_\Omega \leq n$ on $\partial A \cap \partial\Omega$, thanks to (4.32) we find

$$\|H_\Omega - n\|_{L^1(\partial\Omega)} \leq C(n) \left(1 + \|(H_\Omega)^-\|_{L^\infty(\partial\Omega)} \right) \delta(\Omega) \leq C(n, \Lambda) \delta(\Omega). \quad (4.58)$$

Next we claim that

$$\|u\|_{L^p(\mathbb{S}^n)} \leq C(n, p, \Lambda) \delta(\Omega). \quad (4.59)$$

To show this let us assume without loss of generality that

$$\delta(\Omega) \leq \|u\|_{L^p(\mathbb{S}^n)}, \quad (4.60)$$

so that (4.27) and (4.58) imply in particular

$$\|\nabla u\|_{L^p(\mathbb{S}^n)} \leq C_*(n, p, \Lambda) \|u\|_{L^p(\mathbb{S}^n)}. \quad (4.61)$$

Thanks to (4.33), (4.30) holds with $K = C_*(n, p, \Lambda)$, and gives us, taking (4.58) into account,

$$\|u\|_{L^p(\mathbb{S}^n)} \leq C(n, p, \Lambda) \left(\left| \int_{\mathbb{S}^n} u x \right| + \delta(\Omega) \right) \quad (4.62)$$

By (4.6), (4.16), and $\|u\|_{C^1(\mathbb{S}^n)} \leq \varepsilon$

$$\left| \int_{\mathbb{S}^n} u x \right| \leq C(n) \|u\|_{W^{1,2}(\mathbb{S}^n)}^2 \leq C(n) \left(\delta(\Omega)^2 + \|u\|_{C^0(\mathbb{S}^n)} \delta(\Omega) \right) \leq C(n) \varepsilon \delta(\Omega)$$

which combined with (4.62) gives us (4.59). By combining (4.59) with (4.27) and (4.58) we find (4.34). By combining (4.28) and (4.59) we find (4.35). The proof is complete. \square

We now combine the Lemma 4.1, Lemma 4.2 and Proposition 3.2 to prove the estimates in the statement of Theorem 1.5. Their sharpness, which is also part of Theorem 1.5, is addressed in the next section.

Proof of Theorem 1.5, estimates. Let Ω be such that (4.1), (4.2), (4.3) and (4.4) hold, and such that $\partial\Omega$ is smooth. By Lemma 4.1 we find that (1.14), (1.18) and (1.17) hold. We are thus left to prove

$$\max \{ \|u\|_{L^1(\mathbb{S}^n)}, \|(u)^+\|_{C^0(\mathbb{S}^n)} \} \leq C(n) \delta(\Omega).$$

Since $\|u\|_{C^1(\mathbb{S}^n)} \leq \varepsilon(n)$ implies $\delta(\Omega) \leq \delta_0(n)$ for $\delta_0(n)$ as in Theorem 1.2-(ii), we can apply Theorem 1.2-(ii) (at this point we need Ω to be better than $C^{1,1}$ -regular, compare with Proposition 3.2-(iii)) to find an open set E with $C^{1,1}$ -boundary in \mathbb{R}^{n+1} such that

$$\begin{aligned} \Omega &\subset E, & \text{diam}(\Omega) &= \text{diam}(E), \\ |E \setminus \Omega| + \mathcal{H}^n(\partial E \setminus \partial\Omega) &\leq C(n) \delta(\Omega) \\ \|H_E\|_{L^\infty(\partial E)} &\leq n. \end{aligned} \tag{4.63}$$

By (1.18) we have

$$|\Omega \Delta B_1| \leq C(n) \|u\|_{L^1(\mathbb{S}^n)} \leq C(n) \|u\|_{L^2(\mathbb{S}^n)} \leq C(n) \sqrt{\delta(\Omega)}$$

so that by $\Omega \subset E$

$$|E \Delta B_1| \leq |E \setminus \Omega| + |\Omega \Delta B_1| \leq C(n) \sqrt{\delta(\Omega)},$$

We can thus argue as in the proof of Theorem 1.2 and apply Allard's theorem to deduce that, if $\varepsilon(n)$ (and thus $\delta(\Omega)$) is small enough, then for some $v \in C^{1,1}(\mathbb{S}^n)$ we have

$$\partial E = \{ (1 + v(x)) x : x \in \mathbb{S}^n \} \quad \|v\|_{C^1(\mathbb{S}^n)} \leq C(n) \varepsilon(n). \tag{4.64}$$

(Notice that conclusion (ii) in Theorem 1.2 is analogous to (4.64) but holds only after a translation. Here we do not need to translate neither E or Ω , as we know by assumption that Ω is close to B_1 .)

We would now like to apply Lemma 4.2 to E , but the barycenter x_E of ∂E , defined by

$$x_E = \frac{1}{P(E)} \int_{\partial E} x$$

may be non-zero. We notice however that

$$|x_E| \leq C(n) \delta(\Omega). \tag{4.65}$$

Indeed, by (4.64),

$$\begin{aligned} \delta(E) &= P(E) - P(B_1) \leq \mathcal{H}^n(\partial E \cap \partial\Omega) + C(n) \delta(\Omega) - P(B_1) \\ &\leq P(\Omega) + C(n) \delta(\Omega) - P(B_1) \\ &\leq C(n) \delta(\Omega), \end{aligned} \tag{4.66}$$

so that $P(B_1)/2 \leq P(E) \leq 2P(B_1)$ and we can directly focus on the size of $\int_{\partial E} x$. To this end, we first notice that by the assumption $\int_{\Omega} x = 0$, we have

$$\int_{\partial E} x = \int_{\partial E \cap \partial\Omega} x + \int_{\partial E \setminus \partial\Omega} x = \int_{\partial E \setminus \partial\Omega} x - \int_{\partial\Omega \setminus \partial E} x.$$

Now, by (4.63) and (4.64)

$$\left| \int_{\partial E \setminus \partial\Omega} x \right| \leq \|v\|_{C^0(\mathbb{S}^n)} \mathcal{H}^n(\partial E \setminus \partial\Omega) \leq C(n) \delta(\Omega),$$

while

$$\left| \int_{\partial\Omega \setminus \partial E} x \right| \leq \|u\|_{C^0(\mathbb{S}^n)} \mathcal{H}^n(\partial\Omega \setminus \partial E)$$

where $\Omega \subset E \subset A$ (with A the convex envelope of Ω) implies

$$\partial\Omega \setminus \partial E = (\partial\Omega) \cap E \subset (\partial\Omega) \cap A = \partial\Omega \setminus \partial A$$

and where Almgren's identity (1.3) gives $\mathcal{H}^n(\partial\Omega \setminus \partial A) \leq \delta(\Omega)$. Putting everything together, we deduce (4.65).

We can thus apply Lemma 4.2 to $E - x_E$, and find

$$\|v\|_{W^{1,1}(\mathbb{S}^n)} \leq C(n) (\delta(E) + |x_E|), \quad \|v^+\|_{C^0(\mathbb{S}^n)} \leq C(n) (\delta(E) + |x_E|).$$

which combined with (4.65) and (4.66) gives

$$\begin{aligned} \|u\|_{L^1(\mathbb{S}^n)} &\leq C(n) |\Omega \Delta B_1| \leq C(n) (|E \setminus \Omega| + |E \Delta B_1|) \\ &\leq C(n) (\delta(\Omega) + \|v\|_{L^1(\mathbb{S}^n)}) \leq C(n) \delta(\Omega), \end{aligned}$$

that is (1.15). Similarly, since $\Omega \subset E$ we have that

$$\|u^+\|_{C^0(\mathbb{S}^n)} = \sup\{|x| - 1 : x \in \Omega\} \leq \sup\{|x| - 1 : x \in E\} = \|v^+\|_{C^0(\mathbb{S}^n)} \leq C(n) \delta(\Omega),$$

that is (1.16). This completes the proof of the estimates in Theorem 1.5. \square

Proof of Theorem 1.1. Let Ω be a bounded open set with smooth boundary in \mathbb{R}^{n+1} such that $H_\Omega \leq n$ and $\delta(\Omega) \leq \delta(n)$. By Theorem 1.2 there exists an open bounded set E with boundary of class $C^{1,1}$ such that $\Omega \subset E$, $\text{diam}(\Omega) = \text{diam}(E)$, $\|H_E\|_{L^\infty(\partial E)} \leq n$, $|E \setminus \Omega| \leq C(n)\delta(\Omega)$, $\mathcal{H}^n(\partial E \setminus \partial\Omega) \leq C(n)\delta(\Omega)$, and $\partial E = \{(1 + u(x))x : x \in \mathbb{S}^n\}$ where $u \in C^1(\mathbb{S}^n)$ is such that $\|u\|_{C^1(\mathbb{S}^n)} \leq \varepsilon(n)$ for $\varepsilon(n)$ as in Theorem 1.5. In particular, $\|u\|_{L^1(\mathbb{S}^n)} \leq C(n)\delta(E)$ and $\|u^+\|_{C^0(\mathbb{S}^n)} \leq C(n)\delta(E)$. We conclude by arguing as in the last part of the previous proof. \square

5. SHARPNESS OF THEOREM 1.5

The goal of this section is proving the sharpness of Theorem 1.5. Given that the sharpness of (1.15), (1.16), (1.17) (limited to the case $n = 1$) and (1.18) is easily checked by considering the set $\Omega = B_{1+t}$ as $t \rightarrow 0^+$, we focus on proving the sharpness of (1.17) when $n \geq 2$, namely

$$\|u\|_{C^0(\mathbb{S}^n)} \leq C(n) \begin{cases} \delta(\Omega) \log \left(\frac{C(2)}{\delta(\Omega)} \right) & \text{if } n = 2 \\ \delta(\Omega)^{1/(n-1)} & \text{if } n > 2. \end{cases}$$

We are going to do this by constructing a family of open sets with $C^{1,1}$ -boundary $\{\Omega_t\}_{t \in (0, t_0)}$, such that $\partial\Omega_t = \{(1 + u_t(x))x : x \in \mathbb{S}^n\}$ for $u_t \in C^{1,1}(\mathbb{S}^n)$ such that

$$C(n) \|u_t\|_{C^0(\mathbb{S}^n)} \geq \begin{cases} \delta(\Omega_t) \log \left(\frac{1}{\delta(\Omega_t)} \right) & \text{if } n = 2, \\ \delta(\Omega_t)^{1/(n-1)} & \text{if } n > 2. \end{cases} \quad (5.1)$$

For the sake of simplicity we shall just write Ω and u in place of Ω_t and u_t .

We construct $\partial\Omega$ as a surface of revolution obtained by modifying \mathbb{S}^n in the positive cylinder above a small n -dimensional disk. More precisely, we decompose $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, denote by \mathbf{D}_r the ball of radius $r > 0$ centered at the origin in \mathbb{R}^n , and set $\mathbb{S}^{n-1} = \partial\mathbf{D}_1$. We introduce some parameters t , r_0 and r_1 satisfying

$$0 < t < \frac{1}{K(n)} \quad 0 < r_1 < r_0 < \frac{1}{K(n)} \quad (5.2)$$

for a suitably large dimensional constant $K(n)$. Later on r_1 will be specified as a function of n , r_0 , and t . We let

$$\varphi_0(r) = \sqrt{1 - r^2} \quad r \in [0, 1]. \quad (5.3)$$

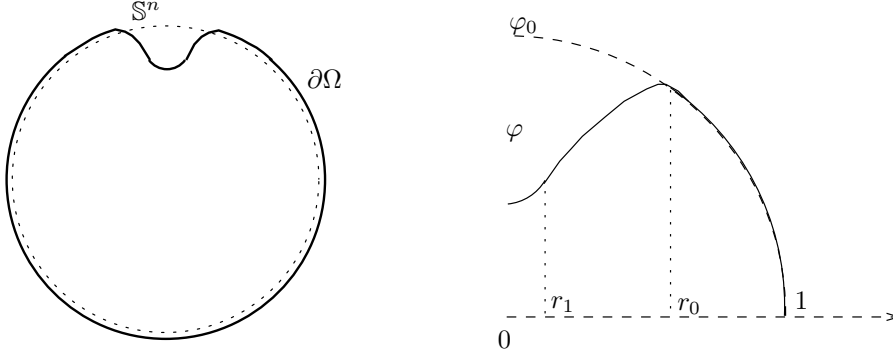


FIGURE 3. The function φ is obtained by carefully joining two circular arcs of opposite curvature. The domain Ω is obtained by slightly scaling out the resulting surface of revolution.

so that $\{(r\omega, \varphi_0(r)) : r \in [0, 1], \omega \in \mathbb{S}^{n-1}\}$ is the unit upper half sphere in \mathbb{R}^{n+1} . We define $\varphi : [0, 1) \rightarrow \mathbb{R}$ by setting

$$\varphi(r) = \begin{cases} \varphi_0(r), & r \in [r_0, 1), \\ \varphi_0(r) - t h(r) & r \in [r_1, r_0], \\ \varphi_0(r_1) - t h(r_1) + \frac{r_1}{\mu} - \sqrt{\left(1 + \frac{1}{\mu^2}\right)r_1^2 - r^2} & r \in [0, r_1), \end{cases} \quad (5.4)$$

where we let $h \in C^2([r_1, r_0])$ be a function such that

$$h(r_0) = h'(r_0) = 0 \quad (5.5)$$

and we define μ in terms of h and r_1 by setting

$$\mu = \varphi'_0(r_1) - t h'(r_1). \quad (5.6)$$

Notice that if $h \in C^2([r_1, r_0])$, then (5.5) and (5.6) guarantee that $\varphi \in C^{1,1}([0, 1])$ with $\varphi'(0) = 0$. We further specify that

$$0 \leq h(r) \leq 1 \quad h'(r) \leq 0 \quad h''(r) \geq 0 \quad \forall r \in [r_1, r_0], \quad (5.7)$$

so that defining S by

$$\begin{aligned} S \cap (\mathbf{D}_{r_0} \times (0, \infty)) &= \{(r\omega, \varphi(r)) : r \in [0, r_0], \omega \in \mathbb{S}^{n-1}\}, \\ S \setminus (\mathbf{D}_{r_0} \times (0, \infty)) &= \partial B_1 \setminus (\mathbf{D}_{r_0} \times (0, \infty)), \end{aligned} \quad (5.8)$$

we find that $S = \partial\Omega^*$ for an open set Ω^* with $C^{1,1}$ -boundary as depicted in Figure 3. Observe that by the definition of φ , $S \cap (\mathbf{D}_{r_1} \times (0, \infty))$ is a spherical cap. By a classical computation, the mean curvature of $\partial\Omega^*$ (as usual computed with respect to ν_{Ω^*}) at the point $r\omega + \varphi(r)e_{n+1}$ corresponding to $\omega \in \mathbb{S}^{n-1}$ and $r \in (0, 1)$ is given by

$$H(r) = H_{\Omega^*}(r\omega + \varphi(r)e_{n+1}) = \frac{-\varphi''(r)}{(1 + \varphi'(r)^2)^{3/2}} - \frac{(n-1)\varphi'(r)}{r\sqrt{1 + \varphi'(r)^2}}. \quad (5.9)$$

Of course, since $\varphi = \varphi_0$ on $(r_0, 1)$, we have $H(r) = n$ for $r \in (r_0, 1)$. Since $\Omega^* \subset B_1$, the boundary $\partial\Omega^*$ is more curved than \mathbb{S}^n near $r = r_0$, and thus we have $H(r) > n$ for r sufficiently close to, and less than, r_0 .

Setting $\Omega = (1+t)\Omega^*$, we claim that for a suitable choice of h , we can achieve

$$H_\Omega \leq n \quad \mathcal{H}^n\text{-a.e. on } \partial\Omega. \quad (5.10)$$

Since $(1+t)H_\Omega = H_{\Omega^*}$, we need

$$H(r) \leq n + nt \quad \forall r \in (r_1, r_0). \quad (5.11)$$

By combining (5.9) and (5.11), we see that finding h amounts to solving the differential inequality

$$H(r) = \frac{-\varphi_0''(r) + t h''(r)}{(1 + (\varphi_0'(r) + t h'(r))^2)^{3/2}} + \frac{(n-1)(-\varphi_0'(r) + t h'(r))}{r \sqrt{1 + (\varphi_0'(r) + t h'(r))^2}} \leq n + tn. \quad (5.12)$$

We will find a solution h which roughly behaves like the fundamental solution for the Laplacian, i.e. like $\log(1/r)$ when $n = 2$ and r^{2-n} when $n > 2$. The precise choice of h is found by considering the Taylor's expansion of (5.12). It is convenient to impose some structural conditions on h in order to control the higher order terms in such expansion. Recalling that $h'(r) \leq 0$ by (5.7), we will require that

$$t |h'(r)| \leq t |h'(r_1)| \leq \frac{1}{K(n)}, \quad (5.13)$$

and since we expect h to behave like the fundamental solution of the Laplacian, we will also require that

$$\max\{|h'(r)|, r |h''(r)|\} \leq K(n) \frac{r_0^n}{r^{n-1}}, \quad (5.14)$$

where recall that $K(n)$ is a large positive constant to be determined. Notice that by (5.6) and (5.13) we definitely have

$$|\mu| \leq \frac{3}{K(n)}. \quad (5.15)$$

Now, let us rewrite the expression of $H(r)$ in (5.12) as

$$H(r) = \left((1-r^2)^{-3/2} + t h''\right) g(r, t)^{-3/2} + (n-1) \left((1-r^2)^{-1/2} + t r^{-1} h'\right) g(r, t)^{-1/2}$$

where

$$g(r, t) = (1-r^2)^{-1} + 2tr(1-r^2)^{-1/2} h'(r) + t^2 h'(r)^2,$$

and observe that by (5.2) and (5.13)

$$|g(r, t) - 1| \leq \frac{r^2}{1-r^2} + \frac{2r}{K(1-r^2)^{1/2}} + \frac{1}{K^2} = \left(\frac{r}{\sqrt{1-r^2}} + \frac{1}{K}\right)^2 \leq \frac{5}{K^2} \quad (5.16)$$

for all $r \in (r_1, r_0)$ and $t \in (0, 1)$. Applying Taylor's theorem

$$f(t) = f(0) + f'(0)t + \int_0^t (t-s) f''(s) ds$$

to $f(t) = g(r, t)^{-k/2}$ for $k = 1, 3$ and using (5.13) and (5.16),

$$\begin{aligned} & \left| g(r, t)^{-k/2} - (1-r^2)^{k/2} + ktr(1-r^2)^{(k+1)/2} h'(r) \right| \\ &= \left| \int_0^t (t-s) h'(r)^2 \left(-k g(r, s)^{-k/2-1} + k(k+2) g(r, s)^{-k/2-2} (r(1-r^2)^{-1/2} + s h'(r))^2 \right) ds \right| \\ &\leq \frac{k+1}{2} t^2 h'(r)^2 \end{aligned}$$

where in the last inequality one choose $K(n)$ large enough to make r_0 and μ (recall (5.15)) sufficiently small. Hence,

$$\begin{aligned} H(r) &\leq \left((1-r^2)^{-3/2} + t h''\right) \left((1-r^2)^{3/2} - 3tr(1-r^2)^2 h' + 2t^2 (h')^2\right) \\ &\quad + (n-1) (1-r^2)^{-1/2} \left((1-r^2)^{1/2} - tr(1-r^2) h' + t^2 (h')^2\right) \\ &\quad + (n-1) tr^{-1} h' \left((1-r^2)^{1/2} - tr(1-r^2) h' - t^2 (h')^2\right) \end{aligned}$$

$$\begin{aligned}
\leq & n + t(1-r^2)^{3/2} h'' + t h' \left(-3r \sqrt{1-r^2} + (n-1)r^{-1}(1-r^2)^{3/2} \right) \\
& + t^2 \left(-3r(1-r^2)^2 h' h'' + 2(1-r^2)^{-3/2} (h')^2 + (n-1)(1-r^2)^{-1/2} (h')^2 \right) \\
& + t^3 \left(2(h')^2 h'' - (n-1)r^{-1} (h')^3 \right).
\end{aligned}$$

Using (5.2) and (5.14),

$$\begin{aligned}
H(r) \leq & n + t(1-r^2)^{3/2} h'' + t h' \left(-3r \sqrt{1-r^2} + (n-1)r^{-1}(1-r^2)^{3/2} \right) \\
& + (n+5)t^2 K^2 r_0^{2n} r^{2-2n} + (n+1)t^3 K^3 r_0^{3n} r^{2-3n}.
\end{aligned}$$

Therefore we can guarantee (5.11) if

$$\begin{aligned}
(1-r^2)^{3/2} h'' + h' \left(-3r \sqrt{1-r^2} + (n-1)r^{-1}(1-r^2)^{3/2} \right) \\
+ (n+5)t K^2 \frac{r_0^{2n}}{r^{2n-2}} + (n+1)t^2 K^3 \frac{r_0^{3n}}{r^{3n-2}} = n.
\end{aligned} \tag{5.17}$$

We will treat the last two terms in (5.17) separately since r^{2-3n} increases faster than r^{2-2n} as $r \downarrow 0$ and thus, as will become apparent below, we need to use the full factor t^2 to control the last term of (5.17). Multiplying both sides by r^{n-1} we get

$$\frac{d}{dr} \left(r^{n-1} (1-r^2)^{3/2} h'(r) \right) = n r^{n-1} - (n+5)t K^2 r_0^{2n} r^{1-n} - (n+1)t^2 K^3 r_0^{3n} r^{1-2n}.$$

Integrating over (r, r_0) and taking (5.5) into account we find that, when $n > 2$

$$-r^{n-1} (1-r^2)^{3/2} h'(r) = r_0^n - r^n - \frac{n+5}{n-2} t K^2 \left(\frac{r_0^{2n}}{r^{n-2}} - r_0^{n+2} \right) - \frac{n+1}{2n-2} t^2 K^3 \left(\frac{r_0^{3n}}{r^{2n-2}} - r_0^{n+2} \right),$$

that is

$$\begin{aligned}
h'(r) = & -(1-r^2)^{3/2} \left(\frac{r_0^n}{r^{n-1}} - r - \frac{n+5}{n-2} t K^2 \left(\frac{r_0^{2n}}{r^{2n-3}} - \frac{r_0^{n+2}}{r^{n-1}} \right) \right. \\
& \left. - \frac{n+1}{2n-2} t^2 K^3 \left(\frac{r_0^{3n}}{r^{3n-3}} - \frac{r_0^{n+2}}{r^{n-1}} \right) \right)
\end{aligned} \tag{5.18}$$

If instead $n = 2$, then

$$h'(r) = -(1-r^2)^{3/2} \left(\frac{r_0^2}{r} - r - 7t K^2 \frac{r_0^4}{r} \log \left(\frac{r_0}{r} \right) - \frac{3}{2} t^2 K^3 \left(\frac{r_0^6}{r^3} - \frac{r_0^4}{r} \right) \right). \tag{5.19}$$

Integrating again over (r, r_0) , and using again (5.5), we find that, if $n > 2$,

$$\begin{aligned}
h(r) = & \int_r^{r_0} (1-s^2)^{3/2} \left(\frac{r_0^n}{s^{n-1}} - s - \frac{n+5}{n-2} t K^2 \left(\frac{r_0^{2n}}{s^{2n-3}} - \frac{r_0^{n+2}}{s^{n-1}} \right) \right. \\
& \left. - \frac{n+1}{2n-2} t^2 K^3 \left(\frac{r_0^{3n}}{s^{3n-3}} - \frac{r_0^{n+2}}{s^{n-1}} \right) \right) ds,
\end{aligned} \tag{5.20}$$

while if $n = 2$, then

$$h(r) = \int_r^{r_0} (1-s^2)^{3/2} \left(\frac{r_0^2}{s} - s - 7t K^2 \frac{r_0^4}{s} \log \left(\frac{r_0}{s} \right) - \frac{3}{2} t^2 K^3 \left(\frac{r_0^6}{s^3} - \frac{r_0^4}{s} \right) \right) ds. \tag{5.21}$$

Having obtained these formulas, we now used them to define h and then check that in this way we obtain the desired family of sets.

More precisely, we argue as follows. For a large dimensional constant $K(n)$, we pick positive parameters t and r_0 so that

$$\frac{t}{r_0} < \frac{1}{K^2} \quad r_0 < \frac{1}{K}. \tag{5.22}$$

(In particular, $t < 1/K^3$.) Next, pick any σ such that

$$\frac{t}{r_0} < \sigma < \frac{1}{K^2}, \quad (5.23)$$

and define r_1 by

$$r_1 = \left(\frac{t}{\sigma}\right)^{1/(n-1)} r_0^{n/(n-1)} \quad \text{so that} \quad t = \frac{r_1^{n-1}}{r_0^n} \sigma. \quad (5.24)$$

This choice of r_1 is motivated by the fact that we will need $t|h'(r_1)| \approx t r_0^n r_1^{1-n} < 1/K$ (recall that $h'(r) \approx r_0^n r^{1-n}$). Notice that r_1 , r_0 and t satisfy (5.2) thanks to (5.22) and (5.24).

Next we define $h \in C^2([r_1, r_0])$ by means of (5.20) if $n > 2$ and of (5.21) if $n = 2$. We claim that (5.5), (5.7), (5.13) and (5.14). Once this is checked, thanks to the above computations and setting $\Omega = (1+t)\Omega^*$ with Ω^* defined thanks to (5.4) and (5.8), we will be able to deduce that Ω is an open set with $C^{1,1}$ -boundary satisfying $H_\Omega \leq n$.

Let us thus check that h satisfies (5.5), (5.7), (5.13) and (5.14). The validity of (5.5) is immediately checked from the definition of h , while the other assertions will follow by showing that

$$t|h'(r_1)| \leq \frac{1}{K}, \quad (5.25)$$

$$-\frac{r_0^n}{r^{n-1}} \leq h'(r) < -\left(1 - \frac{r}{r_0}\right) \frac{r_0^n}{2r^{n-1}}, \quad \forall r \in (r_1, r_0), \quad (5.26)$$

$$0 < h''(r) \leq K \frac{r_0^n}{r^n}, \quad \forall r \in (r_1, r_0). \quad (5.27)$$

For proving (5.25) we just set $r = r_1$ into (5.18) and then, thanks to (5.23) and (5.24), we find that, if $n > 2$,

$$\begin{aligned} t|h'(r_1)| &\leq t \frac{r_0^n}{r_1^{n-1}} + r_1 t + \frac{n+5}{n-2} t^2 K^2 \frac{r_0^{2n}}{r_1^{2n-3}} + \frac{n+1}{2n-2} t^3 K^3 \frac{r_0^{3n}}{r_1^{3-3n}} \\ &= \sigma + \sigma \frac{r_1^n}{r_0^n} + \frac{n+5}{n-2} K^2 \sigma^2 r_1 + \frac{n+1}{2n-2} K^3 \sigma^3 \\ &\leq 3\sigma < \frac{1}{K}. \end{aligned}$$

A similar computation in the case $n = 2$ gives $t|h'(r_1)| \leq 3\sigma \leq 1/K$ if $n = 2$. This proves (5.25). The lower bound in (5.26) follows trivially by (5.18) and (5.19),

$$h'(r) \geq -(1-r^2)^{3/2} \frac{r_0^n}{r^{n-1}} \geq -\frac{r_0^n}{r^{n-1}}.$$

Concerning the fact that $h'(r) < 0$, we notice that by exploiting

$$1 \leq \frac{1 - (r/r_0)^k}{1 - r/r_0} = \sum_{j=0}^{k-1} \left(\frac{r}{r_0}\right)^j \leq k \quad \forall k \in \mathbb{N}, r \in (0, r_0),$$

we find that, if $n > 2$ and thanks to (5.18),

$$\begin{aligned} h'(r) &= -(1-r^2)^{3/2} \left(\frac{r_0^n}{r^{n-1}} \left(1 - \frac{r^n}{r_0^n}\right) - \frac{n+5}{n-2} t K^2 \frac{r_0^{2n}}{r^{2n-3}} \left(1 - \frac{r^{n-2}}{r_0^{n-2}}\right) \right. \\ &\quad \left. - \frac{n+1}{2n-2} t^2 K^3 \frac{r_0^{3n}}{r^{3n-3}} \left(1 - \frac{r^{2n-2}}{r_0^{2n-2}}\right) \right) \\ &\leq -(1-r^2)^{3/2} \frac{r_0^n}{r^{n-1}} \left(1 - \frac{r}{r_0}\right) \left(1 - (n+5) t K^2 \frac{r_0^n}{r^{n-2}} - (n+1) t^2 K^3 \frac{r_0^{2n}}{r^{2n-2}}\right). \end{aligned}$$

Hence by (5.23) and (5.24), and provided K is large enough, we find

$$\begin{aligned} h'(r) &\leq -(1-r_0^2)^{3/2} \frac{r_0^n}{r^{n-1}} \left(1 - \frac{r}{r_0}\right) (1 - (n+5) K^2 \sigma r_1 - (n+1) K^3 \sigma^2) \\ &\leq -\left(1 - \frac{r}{r_0}\right) \frac{r_0^n}{2 r^{n-1}} \end{aligned}$$

where we have used $K^2 r_1 \sigma \leq K^2 r_0 \sigma \leq 1/K$. Similarly, by the concavity of the logarithm ($\log(s) \leq s - 1$ for every $s > 0$) and by definition of r_1 , we find that if $r \in (r_1, r_0)$, then

$$t r_0^2 \frac{\log(r_0/r)}{1 - (r/r_0)} \leq \frac{t r_0^3}{r} \leq t \frac{r_0^3}{r_1} = \sigma r_0 \leq \frac{1}{K^3}$$

while

$$\frac{3}{2} t^2 K^3 \frac{r_0^4}{r^2} \left(1 - \frac{r^2}{r_0^2}\right) \leq \frac{3}{2} t^2 K^3 \frac{r_0^4}{r_1^2} = \frac{3}{2} K^3 \sigma^2 r_0^2 \leq \frac{3}{2K},$$

so that, by (5.19),

$$\begin{aligned} h'(r) &= -(1-r^2)^{3/2} \frac{r_0^2}{r} \left(1 - \frac{r^2}{r_0^2} - 7 t K^2 r_0^2 \log\left(\frac{r_0}{r}\right) - \frac{3}{2} t^2 K^3 \frac{r_0^4}{r^2} \left(1 - \frac{r^2}{r_0^2}\right)\right) \\ &= -(1-r^2)^{3/2} \frac{r_0^2}{r} \left(1 - \frac{r}{r_0}\right) \left(1 + \frac{r}{r_0} - 7 t K^2 r_0^2 \frac{\log(r/r_0)}{1 - (r/r_0)} - \frac{3}{2} t^2 K^3 \frac{r_0^4}{r^2} \left(1 + \frac{r}{r_0}\right)\right) \\ &\leq -(1-r_0^2)^{3/2} \frac{r_0^2}{r} \left(1 - \frac{r}{r_0}\right) \left(1 - \frac{7}{K} - \frac{3}{K}\right) \leq -\left(1 - \frac{r}{r_0}\right) \frac{r_0^2}{2r} \end{aligned}$$

provided K is large enough. This completes the proof of (5.26). We now prove (5.27). We first check the upper bound: by dropping the positive terms on the left-hand side of (5.17) (and using also $h' < 0$ to this end), we find that, since $h'(r) \geq -r_0^n/r^{n-1}$ and $r_0 \leq 1/K$,

$$h''(r) \leq \frac{n}{(1-r^2)^{3/2}} + (n-1) \frac{|h'(r)|}{r} \leq \frac{n}{(1-K^{-2})^{3/2}} + (n-1) \frac{r_0^n}{r^n} \leq 2n \frac{r_0^n}{r^n} \leq K \frac{r_0^n}{r^n},$$

provided K is large enough. Concerning the lower bound in (5.27), we exploit also the upper bound in (5.26) to find

$$\begin{aligned} (1-r^2)^{3/2} h'' &= n - (n-1) (1-r^2)^{3/2} r h' \\ &\quad + 3r \sqrt{1-r^2} h' - (n+5) t K^2 \frac{r_0^{2n}}{r^{2n-2}} - (n+1) t^2 K^3 \frac{r_0^{3n}}{r^{3n-2}} \\ &\geq n + (n-1) (1-r^2)^{3/2} r \left(1 - \frac{r}{r_0}\right) \frac{r_0^n}{2 r^{n-1}} \\ &\quad - 3 \sqrt{1-r^2} \frac{r_0^n}{r^{n-2}} - (n+5) t K^2 \frac{r_0^{2n}}{r^{2n-2}} - (n+1) t^2 K^3 \frac{r_0^{3n}}{r^{3n-2}} \\ &\geq n + \frac{r_0^n}{r^n} \left(\frac{n-1}{4} \left(1 - \frac{r}{r_0}\right) - 3r^2 - (n+5) t K^2 \frac{r_0^n}{r^{n-2}} - (n+1) t^2 K^3 \frac{r_0^{2n}}{r^{2n-2}} \right). \end{aligned}$$

Proceeding from this last inequality, we notice that if $r \in (\tau r_0, r_0)$, $\tau = 1/2$, then by $t < 1/K^3$ we find

$$\begin{aligned} (1-r^2)^{3/2} h''(r) &\geq n - \frac{r_0^n}{r^n} \left(3r^2 + (n+5) t K^2 \frac{r_0^n}{r^{n-2}} + (n+1) t^2 K^3 \frac{r_0^{2n}}{r^{2n-2}} \right) \\ &\geq n - \frac{r^2}{\tau^n} \left(3 + \frac{n+5}{K \tau^{n-2}} + \frac{n+1}{K^3 \tau^{2n-2}} \right) \geq n - \frac{1}{K^2 \tau^n} \left(3 + \frac{n+5}{K \tau^{n-2}} + \frac{n+1}{K^3 \tau^{2n-2}} \right). \end{aligned}$$

By choosing K large enough with respect to n , we find $h''(r) > 0$ for every $r \in (\tau r_0, r_0)$. We now pick $r \in (r_1, \tau r_0)$, and in this case we argue that, thanks to (5.24),

$$\begin{aligned} (1-r^2)^{3/2} h'' &\geq n + \frac{r_0^n}{r^n} \left(\frac{n-1}{4} (1-\tau) - 3r_0^2 - (n+5)t K^2 \frac{r_0^n}{r_1^{n-2}} - (n+1)t^2 K^3 \frac{r_0^{2n}}{r_1^{2n-2}} \right) \\ &= n + \frac{r_0^n}{r^n} \left(\frac{n-1}{4} (1-\tau) - \frac{3}{K^2} - (n+5)K^2 \sigma r_1 - (n+1)\sigma^2 K^3 \right) \geq n, \end{aligned}$$

provided K is large enough with respect to n . We have thus showed that $h'' \geq 0$, thus completing the proof of (5.27).

So far we have proved that if K is a sufficiently large positive dimensional constant, and we use (5.22), (5.23), (5.24), (5.20), (5.21), (5.4), (5.6) and (5.8) to choose r_0 , t , σ and h , and to correspondingly define Ω , then Ω is an open set with $C^{1,1}$ -boundary such that $H_\Omega \leq n$. In this construction t is ranging over the interval $(0, \sigma r_0)$, see (5.23). We now check (5.1).

First note that by (5.6), (5.18), (5.19), and (5.24), μ satisfies

$$\mu = \varphi'_0(r_1) - t h'(r_1) = \sigma - \frac{n+1}{2n-2} K^3 \sigma^3 + O(t^{1/(n-1)}). \quad (5.28)$$

Using $-r_0^n r^{1-n} \leq h'(r) < 0$ and (5.28), we compute that, if $n > 2$,

First we notice that

$$\begin{aligned} P(\Omega^*) - P(B_1) &= \mathcal{H}^{n-1}(\mathbb{S}^{n-1}) \int_0^{r_0} \left(\sqrt{1 + \varphi'(r)^2} - \sqrt{1 + \varphi'_0(r)^2} \right) r^{n-1} dr \\ &\leq C(n) \int_0^{r_0} |\varphi'(r)^2 - \varphi'_0(r)^2| r^{n-1} dr \end{aligned}$$

where we have multiplied and divided by $\sqrt{1 + (\varphi')^2} + \sqrt{1 + (\varphi'_0)^2} \geq 2$. Now, by (5.15) and (5.24)

$$\begin{aligned} \int_0^{r_1} |(\varphi')^2 - (\varphi'_0)^2| r^{n-1} dr &\leq \int_0^{r_1} \left(\frac{1}{(1 + \mu^{-2})r_1^2 - r^2} + \frac{1}{1 - r^2} \right) r^{n+1} dr \\ &\leq \left(\frac{\mu^2}{r_1^2} + 2 \right) \int_0^{r_1} r^{n+1} dr \leq C(n)(\mu^2 r_1^n + r_1^{n+2}) \\ &\leq C(n) r_1^n \leq C(n, \sigma) t^{n/(n-1)}, \end{aligned} \quad (5.29)$$

while the fact that $|h'(r)| \leq r_0^n / r^{n-1}$ for $r \in (r_1, r_0)$ (recall (5.25)) gives

$$\begin{aligned} \int_{r_1}^{r_0} |(\varphi')^2 - (\varphi'_0)^2| r^{n-1} dr &\leq \int_{r_1}^{r_0} \left(\frac{2tr |h'(r)|}{\sqrt{1-r^2}} + t^2 h'(r)^2 \right) r^{n-1} dr \\ &\leq C(n) \int_{r_1}^{r_0} t r_0^n r + t^2 \frac{r_0^{2n}}{r^{n-1}} dr \leq C(n) \left(t r_0^{n+2} + t^2 r_0^{2n} \int_{r_1}^{r_0} r^{1-n} \right). \end{aligned}$$

By definition of r_1 , if $n > 2$ we find

$$t^2 r_0^{2n} \int_{r_1}^{r_0} r^{1-n} \leq C(n) t^2 \frac{r_0^{2n}}{r_1^{n-2}} = C(n) t^{2-(n-2)/(n-1)} r_0^{2n-n(n-2)/(n-1)} \leq C(n) t^{n/(n-1)}.$$

and if $n = 2$ and by $t/r_0 \sigma \leq r_0^2$ (recall (5.23))

$$t^2 r_0^{2n} \int_{r_1}^{r_0} r^{1-n} = t^2 r_0^4 \log(r_0/r_1) = t^2 r_0^4 \log\left(\frac{\sigma}{t r_0}\right) = \sigma t r_0^3 \frac{t r_0}{\sigma} \log\left(\frac{\sigma}{t r_0}\right) \leq \sigma t r_0^3.$$

Summarizing we have proved

$$\int_{r_1}^{r_0} |(\varphi')^2 - (\varphi'_0)^2| r^{n-1} dr \leq \kappa(n) t$$

for a constant $\kappa(n)$ that can be made arbitrarily small by choosing $K(n)$ large enough. By combining this estimate with (5.29) we find

$$P(\Omega^*) - P(B_1) \leq C(n, \sigma) t.$$

Since we can enforce $(1+t)^n \leq 1+2nt$ for every $t < 1/K$, we find

$$\delta(\Omega) = P(\Omega) - P(B_1) = (1+t)^n P(\Omega^*) - P(B_1) \leq 2nt P(\Omega^*) + P(\Omega^*) - P(B_1) \leq C(n, \sigma) t,$$

that is

$$\limsup_{t \rightarrow 0^+} \frac{\delta(\Omega)}{t} \leq C(n, \sigma). \quad (5.30)$$

Next we notice that $\partial\Omega = \{(1+u(x))x : x \in \mathbb{S}^n\}$ for a function u such that

$$\begin{aligned} \|u\|_{C^0(\mathbb{S}^n)} &= \varphi_0(0) - (1+t)\varphi(0) = 1 - (1+t)\varphi(0) \\ &= 1 - (1+t)\sqrt{1-r_1^2} + t(1+t)h(r_1) + (1+t)\left(\sqrt{1+\mu^2}-1\right)\frac{r_1}{\mu} \\ &\geq th(r_1) - t. \end{aligned} \quad (5.31)$$

By (5.20), when $n > 2$ we have

$$th(r_1) = \int_{r_1}^{r_0} \left(t \frac{r_0^n}{s^{n-1}} - ts - \frac{n+1}{2n-2} t^3 K^3 \frac{r_0^{3n}}{s^{3n-3}} \right) ds \quad (5.32)$$

$$\begin{aligned} &+ \int_{r_1}^{r_0} \left((1-s^2)^{3/2} - 1 \right) \left(t \frac{r_0^n}{s^{n-1}} - ts - \frac{n+1}{2n-2} t^3 K^3 \frac{r_0^{3n}}{s^{3n-3}} \right) ds \\ &- \int_{r_1}^{r_0} \frac{n+5}{n-2} t^2 K^2 (1-s^2)^{3/2} \left(\frac{r_0^{2n}}{s^{2n-3}} - \frac{r_0^{n+2}}{s^{n-1}} \right) ds \\ &= I_1 + I_2 + I_3. \end{aligned} \quad (5.33)$$

By (5.24),

$$\begin{aligned} I_1 &= \int_{r_1}^{r_0} \left(t \frac{r_0^n}{s^{n-1}} - ts - \frac{n+1}{2n-2} t^3 K^3 \frac{r_0^{3n}}{s^{3n-3}} \right) ds \\ &= \frac{t r_0^n}{n-2} \left(\frac{1}{r_1^{n-2}} - \frac{1}{r_0^{n-2}} \right) - \frac{t}{2} (r_0^2 - r_1^2) - \frac{(n+1)t^3 K^3 r_0^{3n}}{(2n-2)(3n-4)} \left(\frac{1}{r_1^{3n-4}} - \frac{1}{r_0^{3n-4}} \right) \\ &\geq \left(\frac{t}{\sigma} \right)^{1/(n-1)} r_0^{n/(n-1)} \left(\frac{\sigma}{n-2} - \frac{(n+1)K^3 \sigma^3}{(2n-2)(3n-4)} \right) - C(n, \sigma) t. \end{aligned} \quad (5.34)$$

Using $1 - (1-s^2)^{3/2} \leq (3/2)s^2$ for $s \in (r_1, r_0)$ and also using (5.24), for I_2 we find

$$|I_2| \leq \frac{3}{2} \int_{r_1}^{r_0} \left(t \frac{r_0^n}{s^{n-3}} + \frac{n+1}{2n-2} t^3 K^3 \frac{r_0^{3n}}{s^{3n-5}} \right) ds \quad (5.35)$$

where

$$\begin{aligned} t r_0^n \int_{r_1}^{r_0} \frac{ds}{s^{n-3}} &= \sigma r_1^{n-1} \int_{r_1}^{r_0} \frac{ds}{s^{n-3}} \leq C(n, \sigma) \begin{cases} r_1^2 & \text{if } n = 3 \\ r_1^3 \log(r_0/r_1) & \text{if } n = 4 \\ r_1^3 & \text{if } n > 4 \end{cases} \\ &\leq C(n, \sigma) t^{2/(n-1)}. \end{aligned}$$

and

$$t^3 r_0^{3n} \int_{r_1}^{r_0} \frac{ds}{s^{3n-5}} \leq \sigma^3 r_1^{3(n-1)} \int_{r_1}^{r_0} \frac{ds}{s^{3n-5}} \leq C(n, \sigma) r_1^3 \leq C(n, \sigma) t^{2/(n-1)}.$$

Thus,

$$|I_2| \leq C(n, \sigma) t^{2/(n-1)}. \quad (5.36)$$

Similarly,

$$\begin{aligned} |I_3| &\leq C(n) t^2 r_0^{2n} \int_{r_1}^{r_0} \frac{ds}{s^{2n-3}} ds \leq C(n, \sigma) r_1^{2n-2} \int_{r_1}^{r_0} \frac{ds}{s^{2n-3}} ds \\ &\leq C(n, \sigma) r_1^2 \leq C(n, \sigma) t^{2/(n-1)}. \end{aligned} \quad (5.37)$$

By combining (5.32), (5.34), (5.36), and (5.37) we conclude that if $n > 2$, then

$$\begin{aligned} \|u\|_{C^0(\mathbb{S}^n)} &\geq \left(\frac{t}{\sigma}\right)^{1/(n-1)} r_0^{n/(n-1)} \left(\frac{\sigma}{n-2} - \frac{(n+1)K^3\sigma^3}{(2n-2)(3n-4)}\right) - C(n, \sigma) t^{2/(n-1)} \\ &\geq \frac{t^{1/(n-1)}}{C(n, \sigma)}, \end{aligned} \quad (5.38)$$

up to consider a suitably large value of K , and where we have used $\sigma < 1/K^2$ and $t < 1/K$. We can obtain a similar inequality when $n = 2$. It suffices to notice that, this time starting from (5.21),

$$\begin{aligned} t h(r_1) &= \int_{r_1}^{r_0} \left(\frac{t r_0^2}{s} - ts - \frac{3}{2} t^3 K^3 \left(\frac{r_0^6}{s^3} - \frac{r_0}{s} \right) \right) ds \\ &\quad + \int_{r_1}^{r_0} \left((1-s^2)^{3/2} - 1 \right) \left(\frac{t r_0^2}{s} - ts - \frac{3}{2} t^3 K^3 \left(\frac{r_0^6}{s^3} - \frac{r_0}{s} \right) \right) ds \\ &\quad - 7 \int_{r_1}^{r_0} (1-s^2)^{3/2} t^2 K^2 \frac{r_0^4}{s} \log\left(\frac{r_0}{s}\right) ds = I_1 + I_2 + I_3 \end{aligned}$$

where now using $t r_0^2 = \sigma r_1$ we find

$$\begin{aligned} I_1 &= \int_{r_1}^{r_0} \left(\frac{t r_0^2}{s} - ts - \frac{3}{2} t^3 K^3 \left(\frac{r_0^6}{s^3} - \frac{r_0}{s} \right) \right) ds \geq t r_0^2 \log\left(\frac{\sigma}{r_0 t}\right) - C(n, \sigma) t \\ |I_2| &\leq \frac{3}{2} \int_{r_1}^{r_0} s^2 \left| \frac{t r_0^2}{s} - ts - \frac{3}{2} t^3 K^3 \left(\frac{r_0^6}{s^3} - \frac{r_0}{s} \right) \right| ds \leq C(n, \sigma) t, \end{aligned}$$

while since $s^{-1} \log(r_0/s)$ is decreasing on $s \in (0, r_0)$,

$$|I_3| \leq C(n, \sigma) t^2 r_0^4 \int_{r_1}^{r_0} \frac{1}{s} \log\left(\frac{r_0}{s}\right) ds \leq C(n, \sigma) t^2 r_0^4 \frac{r_0 - r_1}{r_1} \log\left(\frac{r_0}{r_1}\right) \leq C(n, \sigma) t r_0^4 \log\left(\frac{\sigma}{r_0 t}\right).$$

Hence, provided K is large enough,

$$t h(r_1) \geq t \left(r_0^2 - C(n, \sigma) r_0^4 \right) \log\left(\frac{\sigma}{r_0 t}\right) - C(n, \sigma) t \geq \frac{t \log(\sigma/r_0 t)}{C(n, \sigma)}, \quad (5.39)$$

which combined with (5.31) gives us that, if $n = 2$, then

$$\|u\|_{C^0(\mathbb{S}^2)} \geq \frac{t \log(\sigma/r_0 t)}{C(n, \sigma)}. \quad (5.40)$$

By combining (5.30) with (5.38) and (5.40) we complete the proof of (5.1).

6. A SHARP RESULT FOR BOUNDARIES WITH ALMOST CONSTANT MEAN CURVATURE

Here we prove Theorem 1.8 and Theorem 1.10, starting from the latter.

Proof of Theorem 1.10. Let Ω be an open set with $C^{1,1}$ -boundary in \mathbb{R}^{n+1} with $\int_{\partial\Omega} x = 0$ and

$$\partial\Omega = \{(1 + u(x))x : x \in \mathbb{S}^n\}, \quad \|u\|_{C^1(\mathbb{S}^n)} \leq \varepsilon(n)$$

for a function $u \in C^1(\mathbb{S}^n)$. If we let $\varepsilon = \varepsilon(n)$ be as in Lemma 4.1, and we argue as in the first three steps of the proof of Lemma 4.1 (where the assumption $H_\Omega \leq n$ of Lemma 4.1 was not invoked), then, writing $u = a + b \cdot x + R$ as in (4.8), so that

$$a = \frac{1}{\mathcal{H}^n(\mathbb{S}^n)} \int_{\mathbb{S}^n} u \quad b_i = \frac{\int_{\mathbb{S}^n} x_i u}{\int_{\mathbb{S}^n} x_i^2} \quad i = 1, \dots, n,$$

we have the estimates (4.13) and (4.16)

$$(2n+1) \int_{\mathbb{S}^n} R^2 \leq \int_{\mathbb{S}^n} |\nabla R|^2, \quad (6.1)$$

$$|b| \leq C(n) \int_{\mathbb{S}^n} |\nabla R|^2 + \varepsilon O(\|u\|_{W^{1,2}(\mathbb{S}^n)}^2), \quad (6.2)$$

as well as the identities (4.20) and (4.12)

$$H^* = -\operatorname{div}_{\mathbb{S}^n} \left(\frac{\nabla u}{(1+u) \sqrt{(1+u)^2 + |\nabla u|^2}} \right) + \frac{n - \frac{|\nabla u|^2}{(1+u)^2}}{\sqrt{(1+u)^2 + |\nabla u|^2}} \quad (6.3)$$

$$n \mathcal{H}^n(\mathbb{S}^n) a^2 + \int_{\mathbb{S}^n} |\nabla u|^2 - n u^2 = \int_{\mathbb{S}^n} |\nabla R|^2 - n R^2 \quad (6.4)$$

where $H^*(x) = H_{\Omega^*}(x + u(x)x)$ for each $x \in \mathbb{S}^n$. Subtracting n from both sides of (6.3) and multiplying by u , we find that

$$\int_{\mathbb{S}^n} (H^* - n) u = \int_{\mathbb{S}^n} |\nabla u|^2 - n u^2 + \varepsilon O(\|u\|_{W^{1,2}(\mathbb{S}^n)}^2).$$

By (6.4), (6.1) and (6.2), we thus have

$$\begin{aligned} \int_{\mathbb{S}^n} |\nabla R|^2 + R^2 + |b| &\leq C(n) \left(a^2 + \int_{\mathbb{S}^n} (H^* - n) u \right) + \varepsilon O(\|u\|_{W^{1,2}(\mathbb{S}^n)}^2) \\ &\leq C(n) \left(a^2 + \|H_\Omega - n\|_{L^2(\partial\Omega)} \|u\|_{L^2(\mathbb{S}^n)} \right) + \varepsilon O(\|u\|_{W^{1,2}(\mathbb{S}^n)}^2), \end{aligned} \quad (6.5)$$

where we have used $\|H^* - n\|_{L^2(\mathbb{S}^n)} \leq C(n) \|H_\Omega - n\|_{L^2(\partial\Omega)}$. By integrating (6.3) over \mathbb{S}^n after subtracting n from both of its sides, we find that

$$\left| n \int_{\mathbb{S}^n} u - \int_{\mathbb{S}^n} (n - H^*) \right| \leq C(n) \int_{\mathbb{S}^n} u^2 + |\nabla u|^2. \quad (6.6)$$

By the Cauchy-Schwarz inequality

$$\left| \int_{\mathbb{S}^n} (n - H^*) \right| \leq C(n) \|H^* - n\|_{L^2(\mathbb{S}^n)} \leq C(n) \|H_\Omega - n\|_{L^2(\partial\Omega)}.$$

By combining this estimate with (6.6) we find that

$$|a| \leq C(n) \left(\int_{\mathbb{S}^n} u^2 + |\nabla u|^2 + \|H_\Omega - n\|_{L^2(\partial\Omega)} \right).$$

which together with (6.5) gives us

$$\begin{aligned} \int_{\mathbb{S}^n} |\nabla u|^2 + u^2 &\leq C(n) \int_{\mathbb{S}^n} |\nabla R|^2 + R^2 + |b|^2 + a^2 \\ &\leq C(n) \left(\|H_\Omega - n\|_{L^2(\partial\Omega)}^2 + \|H_\Omega - n\|_{L^2(\partial\Omega)} \|u\|_{L^2(\mathbb{S}^n)} \right) + \varepsilon O(\|u\|_{W^{1,2}(\mathbb{S}^n)}^2) \end{aligned}$$

and thus

$$\|u\|_{W^{1,2}(\mathbb{S}^n)} \leq C(n) \|H_\Omega - n\|_{L^2(\partial\Omega)}. \quad (6.7)$$

This proves (1.19). We now prove (1.21). Let us pick p as in (1.20), and notice we can apply (4.26) from Lemma 4.2 to deduce

$$\|u\|_{C^0(\mathbb{S}^n)} \leq C(n, q) \left(\|u\|_{L^2(\mathbb{S}^n)} + \|H_\Omega - n\|_{L^q(\mathbb{S}^n)} \right), \quad \forall q > \frac{n}{2}. \quad (6.8)$$

By setting $q = p$ if $n \geq 4$, or by fixing any $q \in (n/2, 2)$ otherwise, we immediately deduce

$$\|u\|_{C^0(\mathbb{S}^n)} \leq C(n, p) \|H - n\|_{L^p(\mathbb{S}^n)},$$

by combining (1.19) (that is (6.7)), Hölder inequality and (6.8). We conclude the proof of Theorem 1.10 by noticing that if we now assume $\|u\|_{C^{1,\alpha}(\mathbb{S}^n)} \leq K$ for some $\alpha \in (0, 1)$ and $K > 0$, then (1.22) follows immediately by combining (4.31) from Lemma 4.2 with (1.21). \square

Proof of Theorem 1.8. By applying [CM15, Theorem 2.5] while taking into account that $P(\Omega) \leq 2\tau P(B_1)$ we find that, up to a translation setting $\int_{\partial\Omega} x = 0$, $\partial\Omega = \{(1 + u(x))x : x \in \mathbb{S}^n\}$ for a function $u \in C^{1,1}(\mathbb{S}^n)$ such that $\|u\|_{C^{1,1/2}(\mathbb{S}^n)} \leq C(n)$ and $\|u\|_{C^1(\mathbb{S}^n)}$ is arbitrarily small provided $\delta_{\text{cmc}}(\Omega)$ is suitably small. We are thus in the position to apply conclusion (1.22) from Theorem 1.10 to Ω (with the choice $\alpha = 1/2$) to conclude the proof. \square

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